Math 2270 - Exam 3

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (9 Points) *The Four Subspaces*

- (3 points) The nullspace is the orthogonal complement of the ___rowspace___.
- (3 points) If an $m \times n$ matrix has a d-dimensional column space, then its left nullspace has dimension ___$m - d$___.
- (3 points) For an $m \times n$ matrix A calculate:

$$dim(\mathbf{C}(A)) + dim(\mathbf{N}(A)) + dim(\mathbf{C}(A^T)) + dim(\mathbf{N}(A^T)).$$

$$= m + n$$

2. (8 points) *Orthogonal Complements* - Find a basis for the orthogonal complement of the vector space

$$span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}.$$

It will be the nullspace of the matrix

$\begin{pmatrix}1&2&3\\0&1&0\end{pmatrix}$ which is $\boxed{span\left\{\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}}$

3. (20 points) *Projections* - Find the projection of the vector **b** onto the column space of the matrix A:

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}.$$

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad A^+A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

$$A^+\vec{b} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} \qquad A^+A\vec{x} = A^+\vec{b}$$

$$\begin{pmatrix} 3 & 0 & 3 & 5 \\ 0 & 3 & 1 & -1 \\ 3 & 1 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 3 & 5 \\ 0 & 3 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 0 & 3 & 5 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ \frac{1}{3} \end{pmatrix} \Rightarrow \begin{aligned} x_3 &= \frac{1}{2} \\ x_2 &= -\frac{1}{2} \\ x_1 &= \frac{7}{6} \end{aligned} \qquad \vec{x} = \begin{pmatrix} \frac{7}{6} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$A\vec{x} = \vec{p} \Rightarrow \vec{p} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{7}{6} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{13}{6} \\ \frac{7}{6} \\ 0 \\ \frac{10}{6} \end{pmatrix} = \boxed{\frac{1}{6}\begin{pmatrix} 13 \\ 7 \\ 0 \\ 10 \end{pmatrix}}$$

4. (15 points) *Linear Regression* - Find the least squares regression line through the points $(1, 0)$, $(2, 1)$, and $(3, 3)$.

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \hat{\vec{x}} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -\frac{5}{3} \\ x_2 = \frac{3}{2} \end{matrix}$$

$$\boxed{y = -\frac{5}{3} + \frac{3}{2} x}$$

5. (15 points) *Orthonormal Bases* - Find an orthonormal basis for the vector space

Use Gram-Schmidt

$$span\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\end{pmatrix}\right\}.$$

$$\vec{A}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\vec{B}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\frac{(1\,1\,0)\begin{pmatrix}1\\2\\0\end{pmatrix}}{(1\,1\,0)\begin{pmatrix}1\\1\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}\frac{3}{2}\\\frac{3}{2}\\0\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$\vec{C}=\begin{pmatrix}0\\1\\2\end{pmatrix}-\frac{(1\,1\,0)\begin{pmatrix}0\\1\\2\end{pmatrix}}{(1\,1\,0)\begin{pmatrix}1\\1\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\end{pmatrix}-\frac{(-\frac{1}{2}\,\frac{1}{2}\,0)\begin{pmatrix}0\\1\\2\end{pmatrix}}{(-\frac{1}{2}\,\frac{1}{2}\,0)\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}}\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}$$

$$=\begin{pmatrix}0\\1\\2\end{pmatrix}-\begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}-\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}$$

Orthonormal:

$$\vec{q}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix},\quad \vec{q}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix},\quad \vec{q}_3=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

6. (8 points) *Properties of Matrices* - Using the three fundamental properties of the determinant (determinant of the identity is 1, switching rows switches the sign, linearity for each row) prove that a matrix with a row of 0s has determinant 0.

Multiplying the row by t doesn't change the matrix. But, by linearity this multiplies the determinant by t. So,

$$\det(A) = t\det(A)$$

for all $t \in \mathbb{R}$.

$$\Rightarrow \det(A) = 0.$$

7. (15 points) *Calculating Determinants* - Calculate the determinant of the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Use elimination:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 4 & -3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -3 \end{pmatrix} \Rightarrow \det(A) = 1 \times 1 \times -2 \times -3$$

$$= \boxed{6}$$

8. (10 points) *Other Ways of Calculating Determinants* - Calculate the determinant of the matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 0 & 4 \end{pmatrix}$$

using a cofactor expansion along row 3.

$$2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 2(5) + 4(2-9) = 10 - 28 = \boxed{-18}$$