Math 2270 - Practice Exam 3

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (9 Points) *The Four Subspaces*

- (3 points) The rowspace is the orthogonal complement of the nullspace.
- (3 points) If an $m \times n$ matrix has a d-dimensional nullspace, then its rowspace has dimension $n-d$.
- (3 points) For an $m \times n$ matrix A calculate:

$$dim(\mathbf{C}(A)) + dim(\mathbf{N}(A^T)).$$

$= m$

2. (8 points) *Orthogonal Complements* - Find a basis for the orthogonal complement of the vector space

$$span\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}\right\}.$$

It's the nullspace of

$$(1\ 2\ 3) = \boxed{span\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}}$$

3. (20 points) *Projections* - Find the projection of the vector **b** onto the column space of the matrix A:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \qquad = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 2 & 4 \\ 2 & 3 & 2 & 0 \\ 2 & 2 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 2 & 4 \\ 0 & \frac{5}{3} & \frac{2}{3} & -\frac{8}{3} \\ 0 & \frac{2}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 2 & 4 \\ 0 & \frac{5}{3} & \frac{2}{3} & -\frac{8}{3} \\ 0 & 0 & \frac{7}{5} & \frac{7}{5} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & \frac{5}{3} & \frac{2}{3} \\ 0 & 0 & \frac{7}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -\frac{8}{3} \\ \frac{7}{5} \end{pmatrix} \qquad \begin{aligned} x_3 &= 1 \\ x_2 &= -2 \\ x_1 &= 2 \end{aligned}$$

$$\vec{p} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix}}$$

4. (15 points) *Linear Regression* - Find the least squares regression line through the points $(1, 1)$, $(2, 3)$, and $(4, 5)$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 7 & 21 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 27 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 \\ 7 & 21 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 27 \end{pmatrix} \qquad \begin{aligned} 3x_1 + 7x_2 &= 9 \\ 7x_1 + 21x_2 &= 27 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 0 \\ x_2 &= \frac{9}{7} \end{aligned}$$

$$\boxed{y = \frac{9}{7}x + 0}$$

5. (15 points) *Orthonormal Bases* - Find an orthonormal basis for the vector space

$$span\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}.$$

$$\vec{A} = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$\vec{B} = \begin{pmatrix}2\\0\\-2\end{pmatrix} - \frac{(1\ 2\ 0)\begin{pmatrix}2\\0\\-2\end{pmatrix}}{(1\ 2\ 0)\begin{pmatrix}1\\2\\0\end{pmatrix}}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\-2\end{pmatrix} - \frac{2}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\end{pmatrix}$$

$$\vec{C} = \begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{(1\,2\,0)\begin{pmatrix}0\\1\\2\end{pmatrix}}{(1\,2\,0)\begin{pmatrix}1\\2\\0\end{pmatrix}}\begin{pmatrix}1\\2\\0\end{pmatrix} - \frac{(\frac{8}{5}\ -\frac{4}{5}\ -2)\begin{pmatrix}0\\1\\2\end{pmatrix}}{(\frac{8}{5}\ -\frac{4}{5}\ -2)\begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\end{pmatrix}}\begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}\frac{2}{5}\\\frac{4}{5}\\0\end{pmatrix} - \frac{-\frac{24}{\cancel{20}5}}{\frac{180}{25}}\begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}\frac{2}{5}\\\frac{4}{5}\\0\end{pmatrix} + \frac{2}{3}\begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\,6\end{pmatrix} = \begin{pmatrix}\frac{10}{15}\\-\frac{5}{15}\\\frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$$

$$\boxed{\vec{q}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix}, \quad \vec{q}_2 = \frac{\sqrt{5}}{6}\begin{pmatrix}\frac{8}{5}\\-\frac{4}{5}\\-2\end{pmatrix}, \quad \vec{q}_3 = \begin{pmatrix}\frac{2}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{pmatrix}}$$

6. (8 points) *Properties of Matrices* - Using the three fundamental properties of the determinant (determinant of the identity is 1, switching rows switches the sign, linearity for each row) prove that a matrix with a repeated row has determinant 0.

If we switch the rows the matrix is unchanged, but the determinant switches sign. So,

$$\det(A) = -\det(A)$$

$$\Rightarrow \det(A) = 0.$$

7. (15 points) *Calculating Determinants* - Calculate the determinant of the matrix

$$\begin{pmatrix} 1 & -2 & 7 & 9 \\ 3 & -4 & 5 & 5 \\ 3 & 6 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{pmatrix}$$

Use elimination

$$\begin{pmatrix} 1 & -2 & 7 & 9 \\ 3 & -4 & 5 & 5 \\ 3 & 6 & 1 & -1 \\ 4 & 5 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 7 & 9 \\ 0 & 2 & -16 & -22 \\ 0 & 12 & -20 & -28 \\ 0 & 13 & -25 & -34 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 7 & 9 \\ 0 & 2 & -16 & -22 \\ 0 & 0 & 76 & 104 \\ 0 & 0 & 79 & 109 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 7 & 9 \\ 0 & 2 & -16 & -22 \\ 0 & 0 & 76 & 104 \\ 0 & 0 & 0 & \frac{17}{19} \end{pmatrix}$$

$$\Rightarrow \det(A) = 1 \times 2 \times 76 \times \left(\frac{17}{19}\right) = 1 \times 2 \times 4 \times 17$$

$$= \boxed{136}$$

8. (10 points) *Other Ways of Calculating Determinants* - Calculate the determinant of the matrix

$$\begin{pmatrix} 4 & 0 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 5 \end{pmatrix}$$

using a cofactor expansion along column 2.

$$3 \begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} - \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 3(14) - 0$$

$$= \boxed{42}$$