Math 2270 - Exam 2

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 Points) *Subspaces*

Circle the subsets below that are subspaces of $\mathbb{R}^3$:

(a) The set of all vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with $a_3 = a_1 + a_2$.

(b) $\mathit{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(c) The set of all vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with $a_3 \geq a_1 + a_2$.

(d) The set of all vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with $a_3 = 0$.

(e) The set of all vectors $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ with $a_3 = 1$.

2. (20 points) For the following matrix, determine the special solutions for the nullspace, provide a basis for the nullspace, and state the dimension of the nullspace.

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

$$A \xrightarrow[\text{and row 1 from row 3}]{\text{Subtract } 3x \text{ row 1 from row 2}} \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \xrightarrow[\text{row 3}]{\text{Subtract } 2x \text{ row 2 from}} \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

x_2, x_4 free.

$x_2 = 1,\ x_4 = 0.$ $\quad$ $x_2 = 0,\ x_4 = 1.$

$x_3 = 0,\ x_1 = -2$ $\quad$ $x_3 = -1,\ x_1 = -3$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{s}_2 = \begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

The special solutions are a basis, and so the dimension is 2.

3. (20 points) Calculate the rank of the following matrix, provide a basis for the column space, and state the dimension of the column space:

$$B = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix}.$$

$$B \xrightarrow{\text{Add } 2\times \text{ row 1 to row 3}} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{pmatrix} \xrightarrow[\text{add } 3\times \text{ row2 to row 4}]{\text{Subtract row 2 from row3 and}} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}$$

$$\xrightarrow{\text{Multiply row2 by } -1 \text{ and row 4 by } \frac{1}{3}} \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

3 pivots, so $\text{rank}(B) = 3$.

Basis for column space:

$$\begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Dimension of column space is 3.

4. (10 points) For an 8×11 matrix A of rank 5 what are:

- $dim(\mathbf{C}(A)) =$ 5
- $dim(\mathbf{C}(A^T)) =$ 5
- $dim(\mathbf{N}(A)) =$ $11 - 5 = 6$
- $dim(\mathbf{N}(A^T)) =$ $8 - 5 = 3$

5. (25 points) Calculate the complete solution $A\mathbf{x} = \mathbf{b}$ for the following:

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 3 & 1 & -5 & 0 & 8 \\ 1 & 2 & 0 & -5 & -9 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 8 \\ -9 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 5 & 5 \\ 3 & 1 & -5 & 0 & 8 & 8 \\ 1 & 2 & 0 & -5 & -9 & -9 \end{pmatrix} \xrightarrow[\text{and row 1 from row 3}]{\text{Subtract 3x row 1 from row 2}} \begin{pmatrix} 1 & 0 & -2 & 1 & 5 & 5 \\ 0 & 1 & 1 & -3 & -7 & -7 \\ 0 & 2 & 2 & -6 & -14 & -14 \end{pmatrix}$$

$$\xrightarrow{\text{Subtract 2x row 2 from row 3}} \begin{pmatrix} 1 & 0 & -2 & 1 & 5 & 5 \\ 0 & 1 & 1 & -3 & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Last column is free, so solution exists

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ -7 \\ 0 \end{pmatrix}$$

$x_3 = x_4 = x_5 = 0$

$x_2 = -7, \; x_1 = 5$

$$\begin{pmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_3 = 1, \; x_4 = 0, \; x_5 = 0$

$x_2 = -1, \; x_1 = 1$

$x_3 = 0, \; x_4 = 1, \; x_5 = 0$

$x_2 = 3, \; x_1 = -1$

$x_3 = 0, \; x_4 = 0, \; x_5 = 1$

$x_2 = 7, \; x_1 = -5$

Total solution:

$$\boxed{\vec{x} = \begin{pmatrix} 5 \\ -7 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -5 \,|\, 6 \\ 7 \\ 0 \\ 0 \\ 1 \end{pmatrix}}$$

6. (10 points) For what value b_3 does the following equation have a solution? (Don't worry about providing the solution, just give the necessary value of b_3.)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ b_3 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 5 \\ 3 & 3 & 5 & b_3 \end{pmatrix} \xrightarrow{\text{Subtract } 2\times \text{ row1 from row2 and } 3\times \text{ row 1 from row3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -4 & 1 \\ 0 & -3 & -4 & b_3 - 6 \end{pmatrix}$$

$$\xrightarrow{\text{Subtract row 2 from row 3}} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -3 & -4 & 1 \\ 0 & 0 & 0 & b_3 - 7 \end{pmatrix}$$

Must have

$$\boxed{b_3 = 7}$$

7. (10 points extra credit) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for a vector space $\mathbf{V}$ prove that every vector $\mathbf{v} \in \mathbf{V}$ can be expressed *uniquely* as a linear combination of the given basis vectors.

$\vec{V}_1, \ldots, \vec{V}_n$ are a basis for $\vec{V}$ so every vector in $\vec{V}$ can be written as

$$\vec{V} = c_1 \vec{V}_1 + \cdots + c_n \vec{V}_n$$

Suppose

$$\vec{V} = c_1' \vec{V}_1 + \cdots + c_n' \vec{V}_n$$

then

$$\vec{0} = \vec{V} - \vec{V} = (c_1 - c_1')\vec{V}_1 + \cdots + (c_n - c_n')\vec{V}_n$$

As the vectors $\vec{V}_1, \ldots, \vec{V}_n$ are linearly independent we must have

$$c_1 - c_1' = c_2 - c_2' = \cdots = c_n - c_n' = 0.$$

So, $c_1 = c_1'$, $c_2 = c_2'$, $\ldots$, $c_n = c_n'$.

Thus, the representation is unique.