Math 2270 - Practice Exam 2

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 Points) *Subspaces*

   For the following subsets of $\mathbb{R}^n$ explain why they are or are not subspaces of $\mathbb{R}^n$:

   (a) The set of all linear combinations of two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

   (b) The set of all vectors with first component equal to 2.

   (c) The set of all vectors with first component equal to 0.

a) Subspace:

Closed under addition:

$$(c_1\vec{v} + c_2\vec{w}) + (d_1\vec{v} + d_2\vec{w})$$
$$= (c_1+d_1)\vec{v} + (c_2+d_2)\vec{w}$$

Closed under scalar multiplication

$$k(c_1\vec{v} + c_2\vec{w}) = kc_1\vec{v} + kc_2\vec{w}$$

b) Not a subspace. Not closed under addition:

$$\begin{pmatrix}2\\5\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}4\\8\end{pmatrix}$$

c) Subspace:

$$\begin{pmatrix}0\\\vdots\end{pmatrix} + \begin{pmatrix}0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\\vdots\end{pmatrix}$$ closed under addition

$$k\begin{pmatrix}0\\\vdots\end{pmatrix} = \begin{pmatrix}0\\k\cdot\\k\cdot\\k\cdot\end{pmatrix}$$ closed under multiplication

2. (20 points) For the following matrix, determine the special solutions for the nullspace, calculate the nullspace, and give its dimension.

$$A = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{pmatrix}.$$

$$A \xrightarrow[\text{from row2}]{\text{Subtract } 3\times \text{ row 1}} \begin{pmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Special solution 1:

$x_3 = 1$

$x_4 = 0$

$x_2 = 0$

$x_1 = -2$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Special solution 2:

$x_3 = 0$

$x_4 = 1$

$x_2 = -2$

$x_1 = 0$

$$\vec{s}_2 = \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Nullspace} = \text{span} \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\text{dimension} = \boxed{2}$$

3. (20 points) Fow the following matrix, calculate the rank, determine the dimension of the column space, and provide a basis for the column space:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

$$B \xrightarrow[\text{row2}]{\text{Subtract row1 from}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

pivots

2 pivots $\Rightarrow$ $\boxed{\text{rank } 2}$

Dimension of column space is 2.

Basis: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

The columns that become the pivot columns are a basis.

4. (15 points) Calculate the dimension, and find a basis for, the vector space given by:

$$span\left\{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right\}.$$

$$2\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$$

$\begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ are linearly independent.

So, there are 2 basis vectors:

$$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}$$

and the dimension is 2.

5. (30 points) Calculate the complete solution $A\mathbf{x} = \mathbf{b}$ for the following:

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{pmatrix} \xrightarrow{\substack{\text{Subtract } 2\times \\ \text{row1 from row2} \\ \text{and add row1 to row3}}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ & & 6 & 6 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{Subtract} \\ 2\times \text{ row2 from} \\ \text{row 3}}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Divide row2 by 3}} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivots

Free

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$

$x_3 = 1$

$x_1 = -2$

$$\vec{x}_p = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$x_2 = 1$

$x_3 = 0$

$x_1 = -3$

$$\vec{s}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}}$$

Complete solution.

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