Math 2270 - Exam 1

University of Utah

Fall 2012

Name: Key

This is a 50 minute exam. Please show all your work, as a worked problem is required for full points, and partial credit may be rewarded for some work in the right direction.

1. (15 points) *Vector Basics*

   For the vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

   answer the following, or explain why the question does not make sense:

   (a) (3 points) $2\mathbf{a} + 3\mathbf{c} =$

$$2\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 8 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \boxed{\begin{pmatrix} 7 \\ 8 \\ 17 \end{pmatrix}}$$

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(b) (3 points) $||\mathbf{a}|| =$

$$||\vec{a}|| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{4 + 1 + 16}$$

$$= \boxed{\sqrt{21}}$$

(c) (2 points) What are the components of a unit vector in the same direction as **a**?

$$\hat{a} = \begin{pmatrix} \frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(d) (4 points) $\mathbf{b} \cdot \mathbf{c} =$

$$1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = \boxed{6}$$

(e) (3 points) $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c} =$

Does not make sense.
$(\vec{a} \cdot \vec{b})$ is a scalar, and the dot product of a scalar and a vector does not make sense.

2. (10 points) *Matrix Basics*

For the matrices

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 4 & 2 \\ 1 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

answer the following, or explain why the question does not make sense:

(a) (3 points) $A + C =$

$$\begin{pmatrix} 3 & 4 & 2 \\ 2 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 4 & 5 & 3 \\ 2 & 1 & 1 \end{pmatrix}}$$

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 5 \\ 4 & 4 & 2 \\ 1 & 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) (4 points) $CB =$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 5 \\ 4 & 4 & 2 \\ 1 & 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 7 & 5 & 9 \\ 0 & 0 & 0 \end{pmatrix}}$$

(c) (3 points) $BC =$

Does not make sense.

columns of $B \neq$ # rows of C.

3. (15 points) *Elimination Issues*

(a) (5 points) For what value of a in the system of equations below does elimination fail to produce a unique solution?

$$\begin{array}{rcrcl} 3x & + & 2y & = & 10 \\ 6x & + & ay & = & b \end{array}$$

If $\boxed{a=4}$ elimination makes all the coefficients in row 2 on the left 0.

$$0x + 0y = b - 20$$

(b) (5 points) Given the determined value of a, for what value of b are there an infinite number of solutions?

If $b = 20$ elimination gives

$$0x + 0y = 0$$

which is always true.

(c) (5 points) For the determined values of a and b what are two distinct solutions?

$$\boxed{\begin{array}{l} x = 2,\ y = 2 \\ x = 0,\ y = 5 \end{array}}$$

There are, of course, many other possibilities.

4. (20 points) *Systems of Equations*

Use elementary row operations to convert the system of equations

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & 3z & = & 3 \\ 6x & + & 6y & + & 12z & = & 13 \\ 12x & + & 9y & - & z & = & 2 \end{array}$$

into upper-triangular form, and then use back-substitution to solve for the variables x, y, z. Be sure to show all your work.

Subtract 3x row 1 from row 2

$$\begin{array}{l} 2x + 3y + 3z = 3 \\ 0x - 3y + 3z = 4 \\ 12x + 9y - z = 2 \end{array}$$

Subtract 6x row 1 from row 3

$$\begin{array}{l} 2x + 3y + 3z = 3 \\ -3y + 3z = 4 \\ -9y - 19z = -16 \end{array}$$

Subtract 3x row 2 from row3

$$\boxed{\begin{array}{l} 2x + 3y + 3z = 3 \\ -3y + 3z = 4 \\ -28z = -28 \end{array}}$$

upper-triangular form

$$\boxed{\begin{array}{l} z = 1 \\ y = -\frac{1}{3} \\ x = \frac{1}{2} \end{array}}$$

5. (15 points) *Inverting a Matrix*

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 4 & 0 & 1 & 0 \\ 3 & 6 & 5 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 3 & 2 & -3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 3 & -2 \\ 0 & 2 & 0 & -6 & 4 & -2 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & -6 & 4 & -2 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} & -\frac{3}{2} & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right)$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

6. (15 points) *LDU Decomposition*

Find the LDU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ -1 & -4 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & \cancel{0} \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ -1 & -4 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = U$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} = L$$

$$\boxed{\begin{pmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}}$$

7. (10 points) *Symmetric Products*

For the matrix

$$R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$$

(a) (4 points) What is the transpose R^T?

$$\boxed{R^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{pmatrix}}$$

(b) (4 points) What is the symmetric product $R^T R$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 5 & 10 & 3 \\ 10 & 20 & 6 \\ 3 & 6 & 9 \end{pmatrix}}$$

(c) (2 points) Does $R^T R = RR^T$?

No. They're not even the same size.