Math 2270 - Practice Exam 1

University of Utah

Fall 2012

Name: Key

1. (25 points) *Vector Basics*

   For the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

   answer the following:

   (a) (3 points) $\mathbf{a} + \mathbf{b} =$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}}$$

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

(b) (3 points) $2\mathbf{a} =$

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}}$$

(c) (5 points) $||\mathbf{c}|| =$

$$||\vec{c}|| = \sqrt{2^2+1^2+2^2} = \sqrt{9}$$

$$= \boxed{3}$$

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

(d) (5 points) $\mathbf{a} \cdot \mathbf{b} =$

$$1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 = 2 + 8 + 18 = \boxed{28}$$

(e) (5 points) Give the components of a unit vector in the same direction as $\mathbf{b}$.

$$\|\vec{b}\| = \sqrt{2^2 + 4^2 + 6^2}$$

$$= \sqrt{4 + 16 + 36} = \sqrt{56}$$

$$= 2\sqrt{14}$$

$$\hat{b} = \frac{1}{2\sqrt{14}} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}}$$

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

(f) (4 points) Do the three vectors span a line, a plane, or all of $\mathbb{R}^3$?

Plane. $2\vec{a} = \vec{b}$, so those two are linearly dependant. No multiple of $\vec{a}$ equals $\vec{c}$, so $\vec{c}$ is independent of $\vec{a}$ and $\vec{b}$.

2. (15 points) *Matrix Basics*

For the matrices

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

answer the following

(a) (3 points) $A + B =$

Does not exist. Not the same size.

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

(b) (3 points) $A + C =$

$$A + C = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 4 \\ 1 & 5 & 4 \end{pmatrix}$$

(c) (3 points) $AC =$

Does not exist. Wrong sizes

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

(d) (3 points) $AB =$

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 15 & 18 \\ 9 & 17 & 18 \end{pmatrix}$$

(e) (3 points) $BA =$

Does not exist. Wrong sizes.

3. (20 points) *Systems of Equations*

Use elementary row operations to convert the system of equations

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

into upper-triangular form, and then use back-substitution to solve for the variables x, y, z. Be sure to show all your work.

$$\begin{array}{rcl} x - 2y + 3z = 9 & & x - 2y + 3z = 9 \\ y + 3z = 5 & \Rightarrow & y + 3z = 5 \\ -y - z = -1 & & 2z = 4 \end{array}$$

$z = \frac{4}{2} = 2$ $\qquad$ $y + 3(2) = 5 \Rightarrow y = -1$

$x - 2(-1) + 3(2) = 9$

$x + 2 + 6 = 9 \Rightarrow x = 1$

$$\boxed{\begin{array}{l} x = 1 \\ y = -1 \\ z = 2 \end{array}}$$

4. (20 points) *Matrix Form and Inverses*

Write the system of linear equations from the last problem

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

in matrix form, and fine the inverse of the coefficient matrix.

$$\begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 2 & -5 & 5 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 & 0 & \frac{5}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{15}{2} & -\frac{5}{2} & -\frac{9}{2} \\ 0 & 1 & 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 2 & -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \frac{15}{2} & -\frac{5}{2} & -\frac{9}{2} \\ 0 & 1 & 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \frac{15}{2} & -\frac{5}{2} & -\frac{9}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}$$

5. (20 points) *LU Decomposition*

Find the LU decomposition of the coefficient matrix for the system of linear equations

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ -x & + & 3y & & & = & -4 \\ 2x & - & 5y & + & 5z & = & 17 \end{array}$$

$$U = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$ from problem 3.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} = L$$

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{pmatrix}$$