# Math 2270 - Assignment 9

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**Section 4.3** - 1, 3, 9, 10, 12
**Section 4.4** - 2, 6, 11, 18, 21
**Section 5.1** - 1, 3, 8, 13, 24

## 4.3 - Least Squares Approximation

**4.3.1** With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. For the best straight line find its four heights $p_i$ and four errors $e_i$. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 8 \\ 8 & 26 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \end{pmatrix} \qquad \begin{aligned} 4x_1 + 8x_2 &= 36 \\ 8x_1 + 26x_2 &= 112 \end{aligned}$$

$$\Rightarrow \quad 10x_2 = 40 \Rightarrow x_2 = 4$$

$$x_1 = 1$$

So, $\hat{\vec{x}} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$$\vec{p} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} \qquad \vec{e} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}$$



$$E = (-1)^2 + 3^2 + (-5)^2 + 3^2 = \boxed{44}$$ ← Minimum

**4.3.3** Check that $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 3, -5, 3)$ is perpendicular to both columns of $A$. What is the shortest distance $||\mathbf{e}||$ from $\mathbf{b}$ to the column space of $A$?

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix} = -1 + 3 - 5 + 3 = 0$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -5 \\ 3 \end{pmatrix} = 0 + 3 - 15 + 12 = 0$$

$$||\vec{e}|| = \sqrt{(-1)^2 + 3^2 + (-5)^2 + 3^2}$$

$$= \sqrt{44} = 2\sqrt{11}$$

**4.3.9** For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $A\mathbf{x} = \mathbf{b}$ in three unknowns $(C, D, E)$. Set up the three normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ (solution not required).

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix} = A^T A$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}$$

$$A^T A \hat{x} = A^T \vec{b} \Rightarrow \begin{pmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 36 \\ 112 \\ 400 \end{pmatrix}$$

**4.3.10** For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write down the four equations $A\mathbf{x} = \mathbf{b}$. Solve them by elimination. What are $\mathbf{p}$ and $\mathbf{e}$?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

Use elimination

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & 12 & 20 \end{pmatrix} \Rightarrow \begin{aligned} F &= \frac{20}{12} = \frac{5}{3} \\ E &= -\frac{56}{6} = -\frac{28}{3} \\ D &= 8 + \frac{28}{3} - \frac{5}{3} = \frac{47}{3} \\ C &= 0 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} 0 \\ \frac{47}{3} \\ -\frac{28}{3} \\ \frac{5}{3} \end{pmatrix} = \begin{pmatrix} C \\ D \\ E \\ F \end{pmatrix}$$

$$\vec{p} = \vec{b} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

$$\vec{e} = \vec{b} - \vec{p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

**4.3.12** (Recommended) This problem projects $\mathbf{b} = (b_1, \ldots, b_m)$ onto the line through $\mathbf{a} = (1, \ldots, 1)$. We solve $m$ equations $\mathbf{a}x = \mathbf{b}$ in 1 unknown (by least squares).

**(a)** Solve $\mathbf{a}^T\mathbf{a}\hat{x} = \mathbf{a}^T\mathbf{b}$ to show that $\hat{x}$ is the *mean* (the average) of the $\mathbf{b}$'s.

**(b)** Find $\mathbf{e} = \mathbf{b} - \mathbf{a}\hat{x}$ and the *variance* $||\mathbf{e}||^2$ and the *standard deviation* $||\mathbf{e}||$.

**(c)** The horizontal line $\hat{\mathbf{b}} = (3,3,3)$ is closest to $\mathbf{b} = (1,2,6)$. Check that $\mathbf{p} = (3,3,3)$ is perpendicular to $\mathbf{e}$ and find the 3 by 3 projection matrix $P$.

a) $\vec{a}^T\vec{a} = (1\ 1 \cdots 1)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = m$

$\vec{a}^T\vec{b} = (1\ 1 \cdots 1)\begin{pmatrix}b_1\\b_2\\\vdots\\b_m\end{pmatrix} = b_1 + b_2 \cdots + b_m$

$\hat{x} = \dfrac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \dfrac{b_1 + \cdots + b_m}{m}$ ← mean of the $\vec{b}$'s, call it $\bar{b}$.

b) $\vec{e} = \begin{pmatrix}b_1\\\vdots\\b_m\end{pmatrix} - \begin{pmatrix}\bar{b}\\\vdots\\\bar{b}\end{pmatrix} = \begin{pmatrix}b_1 - \bar{b}\\\vdots\\b_m - \bar{b}\end{pmatrix}$

$||\vec{e}||^2 = \sum_{i=1}^{m}(b_i - \bar{b})^2$

$||\vec{e}|| = \sqrt{\sum_{i=1}^{m}(b_i - \bar{b})^2}$



c) $\vec{p}^T\vec{e} = (3,3,3)\begin{pmatrix}-2\\-1\\3\end{pmatrix} = 0$ ✓ $P = \frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

## 4.4 - Orthogonal Bases and Gram-Schmidt

**4.4.2** The vectors $(2, 2, -1)$ and $(-1, 2, 2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Put those into the columns of $Q$ and multiply $Q^TQ$ and $QQ^T$.

$$\vec{q}_1 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} \qquad \vec{q}_2 = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad Q^T = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

$$Q^TQ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$QQ^T = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & -\frac{4}{9} \\ \frac{2}{9} & \frac{8}{9} & \frac{2}{9} \\ -\frac{4}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix}$$

**4.4.6** If $Q_1$ and $Q_2$ are orthogonal matrices, show that their product $Q_1Q_2$ is also an orthogonal matrix. (Use $Q^TQ = I$.)

A matrix $Q$ is orthogonal if and only if $Q^TQ = I$.

$$(Q_1Q_2)^T(Q_1Q_2) = Q_2^TQ_1^TQ_1Q_2$$
$$= Q_2^T I Q_2$$
$$= Q_2^T Q_2 = I \checkmark$$

So, $Q_1Q_2$ is orthogonal.

**4.4.11** **(a)** Gram-Schmidt: Find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in the plane spanned by $\mathbf{a} = (1, 3, 4, 5, 7)$ and $\mathbf{b} = (-6, 6, 8, 0, 8)$.

**(b)** Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

a) $\vec{A} = \vec{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$

$\vec{B} = \vec{b} - \frac{\vec{A}^T \vec{b}}{\vec{A}^T \vec{A}} \vec{A}$

$= \begin{pmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{pmatrix} - \frac{100}{100} \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}$

$$\boxed{\vec{q}_1 = \begin{pmatrix} \frac{1}{10} \\ \frac{3}{10} \\ \frac{4}{10} \\ \frac{5}{10} \\ \frac{7}{10} \end{pmatrix} \quad \vec{q}_2 = \begin{pmatrix} -\frac{7}{10} \\ \frac{3}{10} \\ \frac{4}{10} \\ -\frac{5}{10} \\ \frac{1}{10} \end{pmatrix}}$$

b) $\vec{p} = Q Q^T \vec{b}$

$$= \frac{1}{100} \begin{pmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ -7 & 3 & 4 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{100} \begin{pmatrix} 1 & -7 \\ 3 & 3 \\ 4 & 4 \\ 5 & -5 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -7 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{p} = \frac{1}{100} \begin{pmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{pmatrix}}$$

**4.4.18** (Recommended) Find orthogonal vectors **A**, **B**, **C** by Gram-Schmidt from **a**, **b**, **c**:

$$\mathbf{a} = (1, -1, 0, 0) \qquad \mathbf{b} = (0, 1, -1, 0) \qquad \mathbf{c} = (0, 0, 1, -1).$$

$$\vec{A} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{B} = \vec{b} - \frac{\vec{A}^T \vec{b}}{\vec{A}^T \vec{A}} \vec{A} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} - \frac{(-1)}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{C} = \vec{c} - \frac{\vec{A}^T \vec{c}}{\vec{A}^T \vec{A}} \vec{A} - \frac{\vec{B}^T \vec{c}}{\vec{B}^T \vec{B}} \vec{B}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \frac{-1}{\frac{3}{2}} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 0 \end{pmatrix}$$

$$\vec{A} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \quad \vec{C} = \cancel{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{pmatrix}} \; \cancel{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ +\frac{1}{3} \\ -1 \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{pmatrix}$$

**4.4.21** Find an orthonormal basis for the column space of $A$:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -4 \\ -3 \\ 3 \\ 0 \end{pmatrix}.$$

Then compute the projection of **b** onto that column space.

$$\vec{A}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{A}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 3 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$$

$$\boxed{\vec{q}_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \qquad \vec{q}_2 = \begin{pmatrix} -\frac{5}{2\sqrt{13}} \\ -\frac{1}{2\sqrt{13}} \\ \frac{1}{2\sqrt{13}} \\ \frac{5}{2\sqrt{13}} \end{pmatrix}}$$ ← orthonormal basis.

$$\vec{p} = \begin{pmatrix} \frac{1}{2} & -\frac{5}{2\sqrt{13}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{5}{2\sqrt{15}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{2\sqrt{13}} & -\frac{1}{2\sqrt{13}} & \frac{1}{2\sqrt{13}} & \frac{5}{2\sqrt{13}} \end{pmatrix} \begin{pmatrix} -4 \\ -3 \\ 3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{5}{2\sqrt{13}} \\ \frac{1}{2} & -\frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{1}{2\sqrt{13}} \\ \frac{1}{2} & \frac{5}{2\sqrt{13}} \end{pmatrix} \begin{pmatrix} -2 \\ \sqrt{13} \end{pmatrix} = \begin{pmatrix} -\frac{7}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

## 5.1 - The Properties of Determinants

**5.1.1** If a $4 \times 4$ matrix has $det(A) = \frac{1}{2}$, find $det(2A)$ and $det(-A)$ and $det(A^{-1})$.

$$det(2A) = 2^4 det(A) = 2^3 = \boxed{8}$$

$$det(-A) = (-1)^4 det(A) = det(A) = \boxed{\frac{1}{2}}$$

$$det(A^{-1}) = \frac{1}{det(A)} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

**5.1.3** True of false, with a reason if true and a counterexample if false:

**(a)** The determinant of $I + A$ is $1 + det(A)$.

**(b)** The determinant of $ABC$ is $|A||B||C|$.

**(c)** The determinant of $4A$ is $4|A|$.

**(d)** The determinant of $AB - BA$ is zero. Try an example with $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

a) False. If $A = -I$ then

$$det(I - I) = det(0) = 0,$$

while $1 + det(-I) = 1 + (-1)^n$

which is 2 if $n$ is even. ($A$ is $n \times n$)

b) True. The determinant is multiplicative.

c) False. $\begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} = 16 \neq 4 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 4$

d) False. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} - \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$$

So, $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has $AB - BC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$.

**5.1.8** Prove that every orthogonal matrix ($Q^TQ = I$) has determinant 1 or $-1$.

**(a)** Use the product rule $|AB| = |A||B|$ and the transpose rule $|Q| = |Q^T|$.

**(b)** Use only the product rule. If $|det(Q)| > 1$ then $det(Q^n) = det(Q)^n$ blows up. How do you know this can't happen to $Q^n$?

a) $Q^TQ = I \Rightarrow |Q^TQ| = |I| = 1$

$|Q^TQ| = |Q^T||Q| = |Q|^2$

$\Rightarrow |Q|^2 = 1$ so $|Q| = \pm 1.$

b) Don't do this one. It's wrong. The book wants to argue the determinant of an orthogonal matrix is bounded, but we don't know that a priori.

**5.1.13** Reduce $A$ to $U$ and find $det(A)$ = product of the pivots:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A) = \boxed{1}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

$$\det(A) = 1(-2)\left(-\frac{3}{2}\right) = \boxed{3}$$

**5.1.24** Elimination reduces $A$ to $U$. Then $A = LU$:

$$A = \begin{pmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU.$$

Find the determinants of $L, U, A, U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

$\det(L) = 1$, $\det(U) = -6$

$|A| = |L|\,|U| = (1)(-6) = -6$

$\det(U^{-1}) = \dfrac{1}{\det(U)} = -\dfrac{1}{6}$

$\det(L^{-1}) = \dfrac{1}{1} = 1$

$\Rightarrow \det(U^{-1}L^{-1}) = -\dfrac{1}{6}$

$\det(U^{-1}L^{-1}A) = \det(U^{-1})\det(L^{-1})\det(A)$

$= 1$