Math 2270 - Assignment 9

Dylan Zwick

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Section 4.3 - 1, 3, 9, 10, 12
Section 4.4 - 2, 6, 11, 18, 21
Section 5.1 - 1, 3, 8, 13, 24

4.3 - Least Squares Approximation

4.3.1 With $b = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. For the best straight line find its four heights p_i and four errors e_i. What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?

4.3.3 Check that $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 3, -5, 3)$ is perpendicular to both columns of A. What is the shortest distance $||\mathbf{e}||$ from $\mathbf{b}$ to the column space of A?

4.3.9 For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $A\mathbf{x} = \mathbf{b}$ in three unknowns (C, D, E). Set up the three normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ (solution not required).

4.3.10 For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write down the four equations $A\mathbf{x} = \mathbf{b}$. Solve them by elimination. What are $\mathbf{p}$ and $\mathbf{e}$?

4.3.12 (Recommended) This problem projects $\mathbf{b} = (b_1, \ldots, b_m)$ onto the line through $\mathbf{a} = (1, \ldots, 1)$. We solve m equations $\mathbf{a}x = \mathbf{b}$ in 1 unknown (by least squares).

(a) Solve $\mathbf{a}^T\mathbf{a}\hat{x} = \mathbf{a}^T\mathbf{b}$ to show that $\hat{x}$ is the *mean* (the average) of the $\mathbf{b}$′s.

(b) Find $\mathbf{e} = \mathbf{b} - \mathbf{a}\hat{x}$ and the *variance* $||\mathbf{e}||^2$ and the *standard deviation* $||\mathbf{e}||$.

(c) The horizontal line $\hat{\mathbf{b}} = (3, 3, 3)$ is closest to $\mathbf{b} = (1, 2, 6)$. Check that $\mathbf{p} = (3, 3, 3)$ is perpendicular to $\mathbf{e}$ and find the 3 by 3 projection matrix P.

4.4 - Orthogonal Bases and Gram-Schmidt

4.4.2 The vectors $(2, 2, -1)$ and $(-1, 2, 2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. Put those into the columns of Q and multiply $Q^T Q$ and QQ^T.

4.4.6 If Q_1 and Q_2 are orthogonal matrices, show that their product Q_1Q_2 is also an orthogonal matrix. (Use $Q^TQ = I$.)

4.4.11 **(a)** Gram-Schmidt : Find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in the plane spanned by $\mathbf{a} = (1, 3, 4, 5, 7)$ and $\mathbf{b} = (-6, 6, 8, 0, 8)$.

(b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?

4.4.18 (Recommended) Find orthogonal vectors **A**, **B**, **C** by Gram-Schmidt from **a**, **b**, **c**:

$$\mathbf{a} = (1, -1, 0, 0) \qquad \mathbf{b} = (0, 1, -1, 0) \qquad \mathbf{c} = (0, 0, 1, -1).$$

4.4.21 Find an orthonormal basis for the column space of A:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} -4 \\ -3 \\ 3 \\ 0 \end{pmatrix}.$$

Then compute the projection of **b** onto that column space.

5.1 - The Properties of Determinants

5.1.1 If a 4×4 matrix has $det(A) = \frac{1}{2}$, find $det(2A)$ and $det(-A)$ and $det(A^{-1})$.

5.1.3 True of false, with a reason if true and a counterexample if false:

(a) The determinant of $I + A$ is $1 + det(A)$.

(b) The determinant of ABC is $|A||B||C|$.

(c) The determinant of $4A$ is $4|A|$.

(d) The determinant of $AB - BA$ is zero. Try an example with $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

5.1.8 Prove that every orthogonal matrix $(Q^T Q = I)$ has determinant 1 or -1.

(a) Use the product rule $|AB| = |A||B|$ and the transpose rule $|Q| = |Q^T|$.

(b) Use only the product rule. If $|det(Q)| > 1$ then $det(Q^n) = det(Q)^n$ blows up. How do you know this can't happen to Q^n?

5.1.13 Reduce A to U and find $det(A) =$ product of the pivots:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

5.1.24 Elimination reduces A to U. Then $A = LU$:

$$A = \begin{pmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix} = LU.$$

Find the determinants of $L, U, A, U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.