Math 2270 - Assignment 8

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Fall 2012

Section 4.1 - 6, 7, 9, 21, 24
Section 4.2 - 1, 11, 12, 13, 17

4.1 - Orthogonality of the Four Subspaces

4.1.6 The system of equations $A\mathbf{x} = \mathbf{b}$ has *no solution* (they lead to $0 = 1$):

$$\begin{array}{rcrcrcl} x & + & 2y & + & 2z & = & 5 \\ 2x & + & 2y & + & 3z & = & 5 \\ 3x & + & 4y & + & 5z & = & 9 \end{array}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector $\mathbf{y}$ in which subspace? Its dot product $\mathbf{y}^T\mathbf{b}$ is 1, so no solution $\mathbf{x}$.

$$(x+2y+2z) + (2x+2y+3z) - (3x+4y+5z) = 0$$

$$5 + 5 - 9 = 1$$

So, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\vec{y}$ is in the left-nullspace $\vec{N}(A^T)$.

4.1.7 Every system with no solution is like the one in Problem 4.1.6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called **Frendholm's Alternative**:

Exactly one of these problems has a solution

$$A\mathbf{x} = \mathbf{b} \quad \textbf{OR} \quad A^T\mathbf{y} = \mathbf{0} \quad \text{with} \quad \mathbf{y}^T\mathbf{b} = 1.$$

If $\mathbf{b}$ is not in the column space of A, it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

$$(x_1 - x_2) + (x_2 - x_3) - (x_1 - x_3) = 0$$

$$1 + 1 - 1 = 1.$$

So, $y_1 = 1,\ y_2 = 1,\ y_3 = -1.$

$$\vec{y} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

4.1.9 If $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{0}$. Reason: $A\mathbf{x}$ is in the nullspace of A^T and also in the column space of A and those spaces are orthogonal. *Conclusion: $A^T A$ has the same nullspace as A. This key fact is repeated in the next section.*

4.1.21 Suppose **S** is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $\mathbf{S}^{\perp}$. This is the same as solving $A\mathbf{x} = \mathbf{0}$ for which A?

$$A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} x_3 = 1 \\ x_4 = 0 \end{matrix} \Rightarrow \begin{matrix} x_2 = -1 \\ x_1 = 0 \end{matrix}$$

$$\begin{matrix} x_3 = 0 \\ x_4 = 1 \end{matrix} \Rightarrow \begin{matrix} x_2 = 1 \\ x_1 = -5 \end{matrix}$$

$$\vec{s}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{s}_2 = \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{S}^{\perp} = \vec{N}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4.1.24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

$$\begin{pmatrix} - & \vec{a}_1 & - \\ & \vdots & \\ - & \vec{a}_n & - \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{a}_1^{\,-1} & \cdots & \vec{a}_n^{\,-1} \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

So, $\vec{a}_1^{\,-1}$ is orthogonal to the space spanned by rows 2 through n of A.

4.2 - Projections

4.2.1 Project the vector **b** onto the line through **a**. Check that **e** is perpendicular to **a**:

(a) $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$.

a) $proj_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$

$\vec{a}\cdot\vec{b} = 1+2+2 = 5$

$\vec{a}\cdot\vec{a} = 1+1+1 = 3$

$proj_{\vec{a}}(\vec{b}) = \frac{5}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\vec{e} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 5/3 \\ 5/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

$\vec{a}\cdot\vec{e} = -\frac{2}{3} + \frac{1}{3} + \frac{1}{3} = 0$ ✓.

b) $\vec{a}\cdot\vec{b} = -1-9-1 = -11$

$\vec{a}\cdot\vec{a} = 1+9+1 = 11$

$proj_{\vec{a}}(\vec{b}) = -\frac{11}{11}\begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \vec{b}$!

$\vec{e} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so certainly

$\vec{a}\cdot\vec{e} = 0.$ ✓

4.2.11 Project **b** onto the column space of A by solving $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ and $\mathbf{p} = A\hat{\mathbf{x}}$:

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$.

Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. It should be perpendicular to the columns of A.

a) $A^T = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$A^T\vec{b} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $\quad \hat{\vec{x}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -1 \\ x_2 = 3 \end{matrix}$ $\quad \hat{\vec{x}} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\vec{p} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}$ $\quad \vec{e} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

$(1\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 0$, $\quad (1\ 1\ 0)\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = 0$ ✓

b) $A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ $\quad A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ $\quad A^T\vec{b} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}$

$\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = -2 \\ x_2 = 6 \end{matrix}$ $\quad \hat{\vec{x}} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$ $\quad \vec{p} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 6 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}}$

$\vec{e} = \vec{b} - \vec{p} = \vec{0}$ So certainly $\vec{e}$ is perpendicular to the columns of A.

$\parallel$ b

4.2.12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 4.2.11. Verify that $P_1\mathbf{b}$ gives the first projection $\mathbf{p}_1$. Also verify $P_2^2 = P_2$.

$$P_1 = A(A^TA)^{-1}A^T \qquad A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \vec{p}_1 \ \checkmark$$

$$P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_2^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \checkmark$$

4.2.13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$(A^TA)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P\vec{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$$

P is 4×4.

4.2.17 (*Important*) If $P^2 = P$ show that $(I-P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the left nullspace

$$(I-P)^2 = (I-P)(I-P) = I^2 - IP - PI + P^2$$

$$= I - P - P + P = I - P \checkmark$$

$$(P\vec{x}) \cdot ((I-P)\vec{x})$$

$$= (P\vec{x})^T (I-P)\vec{x} = \vec{x}^T P^T (I-P)\vec{x}$$

$P^T = P$ as P is a projection matrix.

$$= \vec{x}^T P(I-P)\vec{x} = \vec{x}^T (P - P^2)\vec{x}$$

$$= \vec{x}^T (P-P)\vec{x} = \vec{x}^T (\vec{0})\vec{x} = 0.$$

So, $(I-P)\vec{x}$ is orthogonal to $P\vec{x}$. As $P\vec{x}$ is in $\vec{C}(A)$, $(I-P)\vec{x}$ is in $\vec{N}(A^T)$.