Math 2270 - Assignment 8

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Section 4.1 - 6, 7, 9, 21, 24
Section 4.2 - 1, 11, 12, 13, 17

4.1 - Orthogonality of the Four Subspaces

4.1.6 The system of equations $A\mathbf{x} = \mathbf{b}$ has *no solution* (they lead to $0 = 1$):

$$\begin{array}{ccccccc} x & + & 2y & + & 2z & = & 5 \\ 2x & + & 2y & + & 3z & = & 5 \\ 3x & + & 4y & + & 5z & = & 9 \end{array}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector $\mathbf{y}$ in which subspace? Its dot product $\mathbf{y}^T\mathbf{b}$ is 1, so no solution $\mathbf{x}$.

4.1.7 Every system with no solution is like the one in Problem 4.1.6. There are numbers $y_1, \ldots, y_m$ that multiply the m equations so they add up to $0 = 1$. This is called **Frendholm's Alternative**:

Exactly one of these problems has a solution

$$A\mathbf{x} = \mathbf{b} \qquad \textbf{OR} \qquad A^T\mathbf{y} = \mathbf{0} \quad \text{with} \quad \mathbf{y}^T\mathbf{b} = 1.$$

If **b** is not in the column space of A, it is not orthogonal to the nullspace of A^T. Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

4.1.9 If $A^T A\mathbf{x} = \mathbf{0}$ then $A\mathbf{x} = \mathbf{0}$. Reason: $A\mathbf{x}$ is in the nullspace of A^T and also in the ____________ of A and those spaces are ____________. *Conclusion: $A^T A$ has the same nullspace as A. This key fact is repeated in the next section.*

4.1.21 Suppose **S** is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $\mathbf{S}^{\perp}$. This is the same as solving $A\mathbf{x} = \mathbf{0}$ for which A?

4.1.24 Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A?

4.2 - Projections

4.2.1 Project the vector **b** onto the line through **a**. Check that **e** is perpendicular to **a**:

(a) $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix}$.

4.2.11 Project **b** onto the column space of A by solving $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ and $\mathbf{p} = A\hat{\mathbf{x}}$:

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$.

Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. It should be perpendicular to the columns of A.

4.2.12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 4.2.11. Verify that $P_1\mathbf{b}$ gives the first projection $\mathbf{p}_1$. Also verify $P_2^2 = P_2$.

4.2.13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A. What shape is the projection matrix P and what is P?

4.2.17 (*Important*) If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A, $I - P$ projects onto the ____________.