Math 2270 - Assignment 7

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Fall 2012

Section 3.5 - 1, 2, 3, 20, 28
Section 3.6 - 1, 3, 5, 11, 24

3.5 - Independence, Basis, and Dimension

3.5.1 Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are dependent:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \begin{pmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $c_3 = 0$. If $c_3 = 0$ then $c_2 = 0$. If $c_3 = c_2 = 0$ then $c_1 = 0$. So,

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = 0$$

if and only if $c_1 = c_2 = c_3 = 0$. So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent.

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So, $\vec{v}_4 = -\vec{v}_1 - \vec{v}_2 + 4\vec{v}_3$

and therefore $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly dependent.

3.5.2 (Recommended) Find the largest possible number of independent vectors among

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{v}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \mathbf{v}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad \mathbf{v}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

From inspecting rows 2, 3, and 4 we see $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent.

$$\vec{V}_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = V_2 - V_1$$

$$\vec{V}_5 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = V_3 - V_1$$

$$\vec{V}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \vec{V}_3 - \vec{V}_2$$

So, $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6\} = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

So, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis, and the answer is $\boxed{3}$.

3.5.3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

If $a=0$ column 1 is $\vec{0}$, and any set of vectors containing $\vec{0}$ is dependent

If $a \neq 0$ and $d=0$ then

$$\frac{b}{a}\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}$$

So, the columns are dependent.

If $a \neq 0$ and $d \neq 0$ then if $f=0$

$$\begin{pmatrix} c \\ e \\ 0 \end{pmatrix} = \frac{e}{d}\begin{pmatrix} b \\ d \\ 0 \end{pmatrix} + \left(\frac{c}{a} - \frac{be}{ad}\right)\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$$

So, the columns are dependent.

3.5.20 Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbb{R}^3$. Then find a basis for the intersection if that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

The plane $x-2y+3z=0$ is the nullspace of $(1\ -2\ 3)$. So,

$$(1\ -2\ 3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \qquad \begin{matrix} y=1,\ z=0,\ x=2 \\ z=1,\ y=0,\ x=-3 \end{matrix}$$

Basis: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

Intersection with $z=0$. Want nullspace of

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad y=1,\ z=0,\ x=2$$

Basis $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Perpendicular to the plane is the nullspace of $\begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$ which has basis $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

3.5.28 Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

$$\begin{pmatrix} a & b & c \\ -a & -b & -c \end{pmatrix}$$

Basis: $\begin{pmatrix} a & 0 & 0 \\ -a & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b & 0 \\ 0 & -b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & -c \end{pmatrix}$

Basis of subspace:

$$\begin{pmatrix} a & 0 & -a \\ -a & 0 & a \end{pmatrix}, \begin{pmatrix} 0 & b & -b \\ 0 & -b & b \end{pmatrix}.$$

3.6 - Dimension of the Four Subspaces

3.6.1 **(a)** If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?

(b) If a 3 by 4 matrix has rank 3, whare are its column space and left nullspace?

a) $\dim(\vec{C}(A)) = 5$

$\dim(\vec{C}(A^T)) = 5$

$\dim(\vec{N}(A)) = 9-5 = 4$

$\dim(\vec{N}(A^T)) = 7-5 = 2.$

$5+5+4+2 = 16 = 7+9$

b) Column space is a 3-dim subspace of $\mathbb{R}^3$, which must be $\mathbb{R}^3$ itself.

Left nullspace has dimension $3-3=0$, so must just be $\vec{0}$.

3.6.3 Find a basis for each of the four subspaces associated with A:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\vec{C}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$$

$$\vec{C}(A^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\vec{N}(A) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 = 1,\ x_3 = 0,\ x_5 = 0$

$\Rightarrow x_4 = 0,\ x_2 = 0$

$x_1 = 0,\ x_3 = 1,\ x_5 = 0$

$\Rightarrow x_4 = 0,\ x_2 = -2$

$x_1 = 0,\ x_3 = 0,\ x_5 = 1$

$\Rightarrow x_4 = -2,\ x_2 = 2$

$$\vec{N}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$\cancel{\vec{N}(A^T)} = \cancel{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}$

$$\vec{N}(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

3.6.5 If **V** is the subspace spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, find a matrix A that has **V** as its row space. Find a matrix B that has **V** as its nullspace.

$$\boxed{A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}}$$ Easy!

$$\vec{N}(A) \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_3 = 1$, $x_2 = -2$, $x_1 = 1$

$$\vec{N}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

The rowspace and nullspace are orthogonal, so

$$\boxed{B = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}}$$

3.6.11 (Important) A is an m by n matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.

(a) What are all inequalities ($<$ or $\leq$) that must be true between m, n and r?

(b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

a) A cannot have full ~~column~~ row rank. So, $r < m$. In general we must have $r \leq n$. This is it.

b) If A does not have full row rank, then A^T does not have full column rank, and so the dimension of the nullspace of A^T (the left nullspace of A) is positive.

3.6.24 (Important) $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the <u>left nullspace of A</u> contains only the zero vector.

$A^T \vec{y} = \vec{d}$ is solvable when $\vec{d}$ is in the <u>row space</u> of A.