Math 2270 - Assignment 6

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Section 3.3 - 1, 3, 17, 19, 22
Section 3.4 - 1, 4, 5, 6, 18

3.3 - The Rank and Row Reduced Form

3.3.1 Which of these rules gives a correct definition of the *rank* of A?

(a) The number of nonzero rows in R.

(b) The number of columns minus the total number of rows.

(c) The number of columns minus the number of free columns.

(d) The number of 1's in the matrix R.

~~A~~ Only (c) is a correct definition

3.3.3 Find the reduced R for each of these (block) matrices:

$$A=\begin{pmatrix}0&0&0\\0&0&3\\2&4&6\end{pmatrix}\qquad B=\begin{pmatrix}A&A\end{pmatrix}\qquad C=\begin{pmatrix}A&A\\A&0\end{pmatrix}$$

$$A=\begin{pmatrix}0&0&0\\0&0&3\\2&4&6\end{pmatrix}\xrightarrow{\text{Switch rows 1 and 3}}\begin{pmatrix}2&4&6\\0&0&3\\0&0&0\end{pmatrix}\xrightarrow{\text{Divide row 1 by 2, row 2 by 3}}\begin{pmatrix}1&2&3\\0&0&1\\0&0&0\end{pmatrix}$$

$$\xrightarrow{\text{Subtract 3x row 2 from row 1}}\boxed{\begin{pmatrix}1&2&0\\0&0&1\\0&0&0\end{pmatrix}}$$

$$B=\begin{pmatrix}0&0&0&0&0&0\\0&0&3&0&0&3\\2&4&6&2&4&6\end{pmatrix}\xrightarrow{\text{Switch rows 1 and 3}}\begin{pmatrix}2&4&6&2&4&6\\0&0&3&0&0&3\\0&0&0&0&0&0\end{pmatrix}$$

$$\xrightarrow{\text{Divide row 1 by 2, row 2 by 3}}\begin{pmatrix}1&2&3&1&2&3\\0&0&1&0&0&1\\0&0&0&0&0&0\end{pmatrix}\xrightarrow{\text{Subtract 3x row 2 from row 1}}\boxed{\begin{pmatrix}1&2&0&1&2&0\\0&0&1&0&0&1\\0&0&0&0&0&0\end{pmatrix}}$$

$$C=\begin{pmatrix}0&0&0&0&0&0\\0&0&3&0&0&3\\2&4&6&2&4&6\\0&0&0&0&0&0\\0&0&3&0&0&0\\2&4&6&0&0&0\end{pmatrix}\xrightarrow{\text{You get the idea}}\begin{pmatrix}1&2&0&1&2&0\\0&0&1&0&0&1\\0&0&0&0&0&0\\1&2&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\end{pmatrix}$$

3.3.17 (a) Suppose column j of B is a combination of previous columns of B. Show that column j of AB is the same combination of previous columns of AB. Then AB cannot have new pivot columns, so **rank(AB) $\leq$ rank(B)**.

(b) Find A_1 and A_2 so that $rank(A_1B) = 1$ and rank $A_2B = 0$ for $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

a) $B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \end{pmatrix}$ Suppose $\vec{b}_i = c_1\vec{b}_1 + \cdots + c_{i-1}\vec{b}_{i-1}$

$AB = \begin{pmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \end{pmatrix}$ $A\vec{b}_i = A(c_1\vec{b}_1 + \cdots + c_{i-1}\vec{b}_{i-1})$
$= c_1 A\vec{b}_1 + \cdots + c_{i-1}A\vec{b}_{i-1}$

So, $A\vec{b}_i = c_1 A\vec{b}_1 + \cdots + c_{i-1}A\vec{b}_{i-1}$.

b) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A_1B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

Rank 1

$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A_2B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Rank 0

3.3.19 (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \leq rank(A)$ that the rank of A is n. So A is invertible and B must be its two-sided inverse (section 2.5). Therefore $BA = I$ (*which is not so obvious!*).

$AB = I$ $\quad rank(I) = n$. So,

$rank(AB) = n \leq rank(A)$.

As A is $n \times n$ $\quad rank(A) \leq n$.

So, $\quad n \leq rank(A) \leq n \Rightarrow \boxed{rank(A) = n}$

3.3.22 Express A and then B as the sum of two rank one matrices:

$$\textbf{rank} = 2 \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}}$$

3.4 - The Complete Solution to $Ax = \mathbf{b}$.

3.4.1 (Recommended) Execute the six steps of Worked Example **3.4 A** to describe the column space and nullspace of A and the complete solution $A\mathbf{x} = \mathbf{b}$:

$$A = \begin{pmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$$

1. $[A\,\vec{b}] = \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{pmatrix} \xrightarrow{\text{Subtract row 1 from rows 2 and 3}} \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & -1 & -1 & -2 & b_3-b_1 \end{pmatrix}$

$\xrightarrow{\text{Add row 2 to row 3}} \begin{pmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{pmatrix} \xrightarrow{\cancel{\text{Divide row 1 by 2}}} \cancel{\begin{pmatrix} 1 & 2 & 3 & 2 & b_1/2 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$

2. Condition is $b_2 + b_3 - 2b_1 = 0$.

3. The column space of A is spanned by the pivot columns. So, $\vec{C}(A) = \text{span}\left\{\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}\right\}$.
This is a plane in $\mathbb{R}^3$.

4. $x_3 = 1,\ x_4 = 0,\ x_2 = -1,\ x_1 = -1$ $\quad$ $x_3 = 0,\ x_4 = 1,\ x_2 = -2,\ x_1 = 2$

$\vec{s}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ $\vec{s}_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

$\vec{N}(A) = \text{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

6. R $\begin{pmatrix} 1 & 2 & 3 & 2 & b_1/2 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 & \frac{5}{2}b_1 - 2b_2 \\ 0 & 1 & 1 & 2 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_2+b_3-2b_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R$

5. $\vec{x}_p = \begin{pmatrix} \frac{5}{2}b_1 - 2b_2 \\ b_2 - b_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

3.4.4 Find the complete solution (also called the *general solution*) to

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{pmatrix} \xrightarrow{\text{Subtract } 2\times \text{ row 1 from row 2.}} \begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{pmatrix} \xrightarrow{\text{Subtract row 2 from row 3}} \begin{pmatrix} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Particular solution

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$y = t = 0 \quad z = \frac{1}{2} \quad x = \frac{1}{2} \qquad \vec{x}_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

Nullspace

$y = 1,\ t = 0 \quad z = 0 \quad x = -3 \qquad \vec{s}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$y = 0,\ t = 1 \quad z = -2 \quad x = 0 \qquad \vec{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}$

General Solution

All vectors of the form

$$\boxed{\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + c_1\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}}$$

3.4.5 Under what condition on b_1, b_2, b_3 is the system solvable? Include **b** as a fourth column in elimination. Find all solution when that condition holds:

$$\begin{aligned} x &+ 2y - 2z = b_1 \\ 2x &+ 5y - 4z = b_2 \\ 4x &+ 9y - 8z = b_3 \end{aligned}.$$

$$\begin{pmatrix} 1 & 2 & -2 & b_1 \\ 2 & 5 & -4 & b_2 \\ 4 & 9 & -8 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 1 & 0 & b_3 - 4b_1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -2 & b_1 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{pmatrix}$$

Solvable when $b_3 - b_2 - 2b_1 = 0$.

$$\to \begin{pmatrix} 1 & 0 & -2 & 5b_1 - 2b_2 \\ 0 & 1 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{pmatrix}$$

If $b_3 = 2b_1 + b_2$

$$\vec{x}_p = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_3 = 1,\ x_2 = 0,\ x_1 = 2$

$$\vec{s}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The complete solution when $b_3 = 2b_1 + b_2$ is:

$$\boxed{\vec{x} = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$$

3.4.6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find **x** in each case:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{pmatrix} \xrightarrow{\substack{\text{Subtract} \\ 2\times \text{ row 1 from} \\ \text{rows 2 and 3} \\ 3\times \text{ row 1 from row 4}}} \begin{pmatrix} 1 & 2 & b_1 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 3 & b_4-3b_1 \end{pmatrix}$$

$$\xrightarrow{\substack{\text{Subtract} \\ 2\times \text{ row 3 from row 1} \\ \text{and } 3\times \text{ row 3 from row 4}}} \begin{pmatrix} 1 & 0 & 5b_1-2b_3 \\ 0 & 0 & b_2-2b_1 \\ 0 & 1 & b_3-2b_1 \\ 0 & 0 & 3b_1-3b_3+b_4 \end{pmatrix}$$

$$\boxed{\begin{gathered} b_2-2b_1=0 \\ \text{and} \\ 3b_1-3b_3+b_4=0. \end{gathered}}$$

$$\cancel{\vec{x} = \begin{pmatrix} 4b_1-2b_3 \\ 0 \end{pmatrix}}$$

$$\vec{x} = \begin{pmatrix} 5b_1-2b_3 \\ b_3-2b_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & b_1 \\ 2 & 4 & 6 & b_2 \\ 2 & 5 & 7 & b_3 \\ 3 & 9 & 12 & b_4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 3 & 3 & b_4-3b_1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 5b_1-2b_3 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 1 & 1 & b_3-2b_1 \\ 0 & 0 & 0 & 3b_1-3b_3+b_4 \end{pmatrix}$$

Must have
$$\boxed{\begin{gathered} b_2-2b_1=0 \\ 3b_1-3b_3+b_4=0 \end{gathered}}$$

$$\boxed{\vec{x} = \begin{pmatrix} 5b_1-2b_3 \\ b_3-2b_1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}$$

3.4.18 Find by elimination the rank of A and also the rank of A^T:

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{pmatrix} \text{ (rank depends on } q).$$

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 6 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2

$$A^T = \begin{pmatrix} 1 & 2 & -1 \\ 4 & 11 & 2 \\ 0 & 5 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

rank = 2

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{pmatrix}$$

rank = 3 unless $q = 2$, where rank = 2

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & q \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{pmatrix}$$

rank = 3 unless $q = 2$, where rank = 2.