Math 2270 - Assignment 5

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Section 3.2 - 1, 2, 4, 18, 36

3.1 - Spaces of Vectors

In the definition of a vector space, vector addition $\mathbf{x} + \mathbf{y}$ and scalar multiplication $c\mathbf{x}$ must obey the following eight rules:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$

(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$

(3) There is a unique "zero vector" such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x}$

(4) For each $\mathbf{x}$ there is a unique vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

(5) 1 times $\mathbf{x}$ equals $\mathbf{x}$

(6) $(c_1c_2)\mathbf{x} = c_1(c_2\mathbf{x})$

(7) $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$

(8) $(c_1 + c_2)\mathbf{x} = c_1\mathbf{x} + c_2\mathbf{x}$.

3.1.1 Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $c\mathbf{x} = (cx_1, cx_2)$, which of the eight conditions are not satisfied?

$$\vec{x} + \vec{y} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$$

$$\vec{y} + \vec{x} = (y_1, y_2) + (x_1, x_2) = (y_1 + x_2, y_2 + x_1)$$

$$\vec{x} + \vec{y} \neq \vec{y} + \vec{x}$$

So, condition 1 is not satisfied.

$$\vec{x} + (\vec{y} + \vec{z}) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2))$$

$$= (x_1, x_2) + (y_1 + z_2, y_2 + z_1)$$

$$= (x_1 + y_2 + z_1, x_2 + y_1 + z_2)$$

$$(\vec{x} + \vec{y}) + \vec{z} = (x_1 + y_2, x_2 + y_1) + (z_1, z_2)$$

$$= (x_1 + y_2 + z_2, x_2 + y_1 + z_1) \neq \vec{x} + (\vec{y} + \vec{z}).$$

So, condition 2 is not satisfied.

3.1.2 Suppose the multiplication $c\mathbf{x}$ is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2). With the usual addition in $\mathbb{R}^2$ are the eight conditions satisfied?

$$1\vec{x} = 1(x_1, x_2) = (1x_1, 0) = (x_1, 0)$$

So, condition 5 is not satisfied.

the rest are satisfied.

3.1.10 Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

(b) The plane of vectors with $b_1 = 1$.

(c) The vectors with $b_1 b_2 b_3 = 0$.

(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.

(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.

(f) All vectors with $b_1 \leq b_2 \leq b_3$.

a) Subspace

b) Not a subspace

c) Not a subspace

d) Subspace

e) Subspace

f) Not a subspace

3.1.20 For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

a) $$\begin{pmatrix} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_1+b_3 \end{pmatrix}$$

So, $\boxed{b_2 = 2b_1 \text{ and } b_3 = -b_1}$

b) $$\begin{pmatrix} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & b_1 \\ 0 & 1 & b_2-2b_1 \\ 0 & 0 & b_1+b_3 \end{pmatrix}$$

$\boxed{b_3 = -b_1}$

3.1.23 ~~If we~~ add an extra column **b** to a matrix A, then the column space gets larger unless ____________. Give an example where the column space gets larger and an example where it doesn't. Why is $A\mathbf{x} = \mathbf{b}$ solvable exactly when the column space *doesn't* ger larger - it is the same for A and $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$?

$\vec{b}$ is in the column space of A.

Example where the column space gets larger:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example where the column space does not get larger:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The possible outputs of $A\vec{x}$ are exactly those vectors in the column space of A. So, if $A\vec{x} = \vec{b}$ is solvable, then $\vec{b}$ is in the column space of A, and the column space does not get larger when we add $\vec{b}$.

3.2 - The Nullspace of A: Solving $A\mathbf{x} = \mathbf{b}$

3.2.1 Reduce the matrices to their ordinary echelon forms U:

(a) $A = \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix}$.

a) $\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{subtract row 1 from row 2}} \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}$

$\xrightarrow{\text{subtract row2 from row3}} \begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{pmatrix} \xrightarrow{\text{subtract 2x row 2 from row 3.}} \begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

3.2.2 For the matrices in Problem 3.2.1, find a special solution for each free variable. (Set the free variable equal to 1. Set the other free variables equal to zero.)

a) $$\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

Pivot columns are columns 1 and 3. Free columns are columns 2, 4, 5. So, free variables are x_2, x_4, x_5.

$\cancel{x_2 = 0}$ $x_2 = 1, x_4 = 0, x_5 = 0.$

$x_3 = 0 \quad x_1 = -2 \qquad \vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$x_4 = 1, x_2 = 0, x_5 = 0$

$x_3 = -2 \quad x_1 = 0 \qquad \vec{s}_2 = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

$x_5 = 1, x_2 = 0, x_4 = 0$

$x_3 = -3 \quad x_1 = 0 \qquad \vec{s}_3 = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

b) $$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Pivot variables: x_1, x_2

Free variable: x_3

$x_3 = 1, x_2 = -1, x_1 = 1 \qquad \vec{s}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

3.2.4 By further row operations on each U in Problem 3.2.1, find the reduced echelon form R. *True or false*: The nullspace of R equals the nullspace of U.

a)
$$\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{\text{subtract}\\ \text{row 2}\\ \text{from row 1.}}} \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{\text{Subtract}\\ \text{row 2 from}\\ \text{row 1}}} \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\substack{\text{Multiply row 1}\\ \text{by } \frac{1}{2}, \text{ and row 2}\\ \text{by } \frac{1}{4}.}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

<u>True</u>. In general $\vec{N}(A) = \vec{N}(U) = \vec{N}(R)$.

3.2.18 The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 3.2.17. One particular point on this plane is $(12, 0, 0)$. All points on the plan have the form (fill in the first components)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$x = 12 + 3y + z$

3.2.36 How is the nullspace $\mathbf{N}(C)$ related to the spaces $\mathbf{N}(A)$ and $\mathbf{N}(B)$, if $C = \begin{pmatrix} A \\ B \end{pmatrix}$?

$$C\vec{x} = \begin{pmatrix} A \\ B \end{pmatrix}\vec{x} = \begin{pmatrix} A\vec{x} \\ B\vec{x} \end{pmatrix} = \vec{0} \text{ if and only if } A\vec{x} = B\vec{x} = \vec{0}.$$

So, $\vec{N}(C)$ is all vectors in both $\vec{N}(A)$ and $\vec{N}(B)$. Written mathematically

$$\boxed{\vec{N}(C) = \vec{N}(A) \cap \vec{N}(B)}$$