Math 2270 - Assignment 4

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Section 2.5 - 1, 7, 25, 27, 29
Section 2.6 - 3, 5, 7, 13, 16
Section 2.7 - 1, 12, 19, 22, 40

2.5 - Inverse Matrices

2.5.1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{-12}\begin{pmatrix} 0 & -3 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{12} \\ \frac{4}{12} & 0 \end{pmatrix}$$

$$B^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$C^{-1} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$$

2.5.7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\mathbf{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

(c) What happens to row 3 in elimination?

a) $$A\vec{x} = \begin{pmatrix} (\text{row } 1 \text{ of } A)\cdot\vec{x} \\ (\text{row } 2 \text{ of } A)\cdot\vec{x} \\ (\text{row } 3 \text{ of } A)\cdot\vec{x} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

If (row 3 of A) = (row 1 of A) + (row 2 of A)

then $\Rightarrow$ (row 3 of A)$\cdot\vec{x}$ = (row 1 of A)$\cdot\vec{x}$ + (row 2 of A)$\cdot\vec{x}$

= (row 1 of A)$\cdot\vec{x}$ as (row 2 of A)$\cdot\vec{x} = 0$.

So, we'd have to have

(row 1 of A)$\cdot\vec{x} = 1$ and

(row 1 of A)$\cdot\vec{x} = 0$

which is impossible.

b) If $b_3 = b_1 + b_2$ then there could be a solution

c) It becomes a row of 0s.

2.5.25 Find A^{-1} and B^{-1} (*if they exist*) by elimination of $\begin{bmatrix} A & I \end{bmatrix}$ and $\begin{bmatrix} B & I \end{bmatrix}$:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & -1 \\ 0 & \frac{3}{2} & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{4}{3} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{3}{2} & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \boxed{A^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 2 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Fail. $\boxed{\text{B is not invertible.}}$

2.5.27 Invert these matrices A by the Gauss-Jordan method starting with $[\ A\ \ I\ ]$:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \boxed{A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \qquad \boxed{A^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}}$$

2.5.29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.
(b) Every matrix with 1's down the main diagonal is invertible.
(c) If A in invertible then A^{-1} and A^2 are invertible.

a) True. Elimination must fail with a row of 0s

b) False. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible.

c) True. The inverse of A^{-1} is A.

The inverse of A^2 is

$(A^{-1})(A^{-1}) = A^{-2}$.

2.6 - Elimination = Factorization: $A = LU$

2.6.3 Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a triangular $U\mathbf{x} = \mathbf{c}$:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & + & 2y & + & 3z & = & 7 \\ x & + & 3y & + & 6z & = & 11 \end{array}$$

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ & & y & + & 2z & = & 2 \\ & 7 & 2y & + & 5z & = & 6 \end{array}$$

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 2 \end{array}$$

The equation $z = 2$ in $U\mathbf{x} = \mathbf{c}$ comes from the original $x+3y+6z = 11$ in $A\mathbf{x} = \mathbf{b}$ by subtracting $\ell_{31} =$ ____1____ times equation 1 and $\ell_{32} =$ ____2____ times the *final* equation 2. Reverse that to recover $\begin{bmatrix} 1 & 3 & 6 & 11 \end{bmatrix}$ in the last row of A and $\mathbf{b}$ from the final $\begin{bmatrix} 1 & 1 & 1 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$ in U and $\mathbf{c}$:

$$\text{Row 3 of } \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = (\ell_{31} \text{ Row } 1 + \ell_{32} \text{ Row } 2 + 1 \text{ Row } 3) \text{ of } \begin{bmatrix} U & \mathbf{c} \end{bmatrix}.$$

In matrix notation this is multiplication by L. So $A = LU$ and $\mathbf{b} = L\mathbf{c}$.

2.6.5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = L$$

$$A = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{pmatrix}}$$

2.6.7 What three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into L times U:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \qquad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}.$$

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}$$

$$E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix}$$

$$E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$E_{21}^{-1}E_{31}^{-1}E_{22}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} = L$$

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

2.6.13 (*Recommended*) Compute L and U for the symmetric matrix A:

$$A = \begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

$$\begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{pmatrix} \Rightarrow \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{pmatrix} = U}$$

$$\boxed{L = \begin{pmatrix} +1 & 0 & 0 & 0 \\ +1 & +1 & 0 & 0 \\ +1 & +1 & +1 & 0 \\ +1 & +1 & +1 & +1 \end{pmatrix}}$$

$$\begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{pmatrix}$$

$\boxed{\text{Must have } a \neq 0,\ b \neq a,\ c \neq b,\ d \neq c}$

2.6.16 Solve $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$. *What was A*?

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and}$$

$$\mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \begin{aligned} c_1 &= 4 \\ c_2 &= 1 \\ c_3 &= 1 \end{aligned}$$

$$\boxed{\vec{c} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{aligned} x_3 &= 1 \\ x_2 &= 0 \\ x_1 &= 3 \end{aligned}$$

$$\boxed{\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}}$$

2.7 - Transposes and Permutations

2.7.1 Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix} \qquad \text{and also} \qquad A = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & 9 \\ 0 & 3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 \\ -9 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & \frac{1}{3} \end{pmatrix}$$

$$(A^{-1})^T = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix} \quad (A^T)^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & \frac{1}{3} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}$$

$$A^{-1} = \frac{1}{0-c^2}\begin{pmatrix} 0 & -c \\ -c & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & -\frac{1}{c^2} \end{pmatrix}$$

$$(A^{-1})^T = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & -\frac{1}{c^2} \end{pmatrix} \quad (A^T)^{-1} = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & -\frac{1}{c^2} \end{pmatrix}$$

2.7.12 Explain why the dot product of **x** and **y** equals the dot product of $P\mathbf{x}$ and $P\mathbf{y}$. Then from $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ deduce that $P^TP = I$ for any permutation. With $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, 4, 2)$ choose P to show that $P\mathbf{x} \cdot \mathbf{y}$ is not always $\mathbf{x} \cdot P\mathbf{y}$.

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Permuting the order doesn't change the sum.

$$(P\vec{x})^T(P\vec{y}) = \vec{x}^T P^T P \vec{y} = \vec{x}^T \vec{y}$$

So, $P^T P = I$.

$$\vec{x} \cdot \vec{y} = 1 + 8 + 6 = 15$$

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad P\vec{x} = (3,2,1) \quad P\vec{y} = (2,4,1)$$

$$P\vec{x} \cdot \vec{y} = 3 + 8 + 2 = 13$$

$$\vec{x} \cdot P\vec{y} = (1,2,3) \cdot (2,4,1) = 13$$

$$\boxed{P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \qquad P\vec{x} = (2,3,1) \quad P\vec{y} = (4,2,1)$$

$$P\vec{x} \cdot \vec{y} = 2 + 12 + 2 = 16$$

$$\vec{x} \cdot P\vec{y} = 4 + 4 + 3 = \cancel{16}\ \cancel{13}\ 11$$

2.7.19 Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^T AR$ to show its symmetry. What shape is this matrix?

(b) Show why $R^T R$ has no negative numbers on its diagonal.

a) $(R^T A R)^T = R^T A^T (R^T)^T = R^T A^T R$

$= R^T A R$ as A is symmetric.

$(n \times m) \times (m \times m) \times (m \times n)$ gives an $\boxed{n \times n}$ matrix

b) $R^T R = A$. The diagonal term a_{ii} is

$a_{ii} = (\text{Row } i \text{ of } R^T) \cdot (\text{Column } i \text{ of } R)$

$= (\text{Column } i \text{ of } R) \cdot (\text{Column } i \text{ of } R)$

≥ 0.

as $\vec{v} \cdot \vec{v} \geq 0$ for any vector $\vec{v}$.

2.7.22 Find the $PA = LU$ factorization (and check them) for

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} = L$$

$$\boxed{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

2.7.40 Suppose Q^T *equals* Q^{-1} (transpose equals inverse, so $Q^TQ = I$):

(a) Show that the columns $q_1, \ldots, q_n$ are unit vectors: $||\mathbf{q}_i||^2 = 1$.

(b) Show that every two columns of Q are perpendicular: $\mathbf{q}_1^T\mathbf{q}_2 = 0$.

(c) Find a 2 by 2 example with first entry $q_{11} = \cos\theta$.

a) The entry $(Q^TQ)_{ij}$ = (row i of Q^T)$\cdot$(column j of Q)

$= ($~~row~~ column i of $Q)\cdot($column j of $Q)$

$= \vec{q}_i \cdot \vec{q}_j = \delta_{ij}$.

For $i=j$ we get $\vec{q}_i \cdot \vec{q}_i = 1$

So, $||\vec{q}_i|| = 1$.

b) $\vec{q}_i \cdot \vec{q}_j = (Q^TQ)_{ij} = 0$ if $i \neq j$.

c) $$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$