Math 2270 - Assignment 4

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Section 2.5 - 1, 7, 25, 27, 29
Section 2.6 - 3, 5, 7, 13, 16
Section 2.7 - 1, 12, 19, 22, 40

2.5 - Inverse Matrices

2.5.1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C:

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$$

2.5.7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

(a) Explain why $A\mathbf{x} = (1, 0, 0)$ cannot have a solution.

(b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

(c) What happens to row 3 in elimination?

2.5.25 Find A^{-1} and B^{-1} (*if they exist*) by elimination of $\begin{bmatrix} A & I \end{bmatrix}$ and $\begin{bmatrix} B & I \end{bmatrix}$:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

2.5.27 Invert these matrices A by the Gauss-Jordan method starting with $\begin{bmatrix} A & I \end{bmatrix}$:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

2.5.29 True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If A in invertible then A^{-1} and A^2 are invertible.

2.6 - Elimination = Factorization: $A = LU$

2.6.3 Forward elimination changes $A\mathbf{x} = \mathbf{b}$ to a triangular $U\mathbf{x} = \mathbf{c}$:

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ x & + & 2y & + & 3z & = & 7 \\ x & + & 3y & + & 6z & = & 11 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ & & y & + & 2z & = & 2 \\ & & 2y & + & 5z & = & 6 \end{array}$$

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 5 \\ & & y & + & 2z & = & 2 \\ & & & & z & = & 2 \end{array}$$

The equation $z = 2$ in $U\mathbf{x} = \mathbf{c}$ comes from the original $x+3y+6z = 11$ in $A\mathbf{x} = \mathbf{b}$ by subtracting $\ell_{31} =$ ____________ times equation 1 and $\ell_{32} =$ ____________ times the *final* equation 2. Reverse that to recover $\begin{bmatrix} 1 & 3 & 6 & 11 \end{bmatrix}$ in the last row of A and $\mathbf{b}$ from the final $\begin{bmatrix} 1 & 1 & 1 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix}$ in U and $\mathbf{c}$:

Row 3 of $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ = (ℓ_{31} Row 1 + ℓ_{32} Row 2 + 1 Row 3) of $\begin{bmatrix} U & \mathbf{c} \end{bmatrix}$.

In matrix notation this is multiplication by L. So $A = LU$ and $\mathbf{b} = L\mathbf{c}$.[1]

[1] This problem is a bit verbose. The only thing you're being asked to do is fill in the blanks.

2.6.5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{pmatrix}$$

2.6.7 What three elimination matrices E_{21}, E_{31}, E_{32} put A into its upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1}, E_{31}^{-1} and E_{21}^{-1} to factor A into L times U:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \qquad\qquad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}.$$

2.6.13 (*Recommended*) Compute L and U for the symmetric matrix A:

$$A = \begin{pmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{pmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

2.6.16 Solve $L\mathbf{c} = \mathbf{b}$ to find $\mathbf{c}$. Then solve $U\mathbf{x} = \mathbf{c}$ to find $\mathbf{x}$. *What was* A?

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

2.7 - Transposes and Permutations

2.7.1 Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{pmatrix} 1 & 0 \\ 9 & 3 \end{pmatrix} \qquad \text{and also} \qquad A = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}.$$

2.7.12 Explain why the dot product of $\mathbf{x}$ and $\mathbf{y}$ equals the dot product of $P\mathbf{x}$ and $P\mathbf{y}$. Then from $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ deduce that $P^TP = I$ for any permutation. With $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, 4, 2)$ choose P to show that $P\mathbf{x} \cdot \mathbf{y}$ is not always $\mathbf{x} \cdot P\mathbf{y}$.

2.7.19 Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^T AR$ to show its symmetry. What shape is this matrix?

(b) Show why $R^T R$ has no negative numbers on its diagonal.

2.7.22 Find the $PA = LU$ factorization (and check them) for

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

2.7.40 Suppose Q^T *equals* Q^{-1} (transpose equals inverse, so $Q^T Q = I$):

(a) Show that the columns $q_1, \ldots, q_n$ are unit vectors: $||\mathbf{q}_i||^2 = 1$.

(b) Show that every two columns of Q are perpendicular: $\mathbf{q}_1^T \mathbf{q}_2 = 0$.

(c) Find a 2 by 2 example with first entry $q_{11} = \cos\theta$.