Math 2270 - Assignment 2

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Section 2.1 - 4,5,9,13,17
Section 2.2 - 3,6,11,12,19

1 Section 2.1 - Vectors and Linear Equations

2.1.4 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$. Find a third point halfway between.

$z = 2$

$x + y + 3(2) = 6$

$x - y + 2 = 4$

$x + y = 0$

$x - y = 2$

$2x = 2$

$x = 1$

$y = -1$

$\boxed{(1, -1, 2)}$

~~(1, -1, 2)~~

$z = 0$

$x + y = 6$

$x - y = 4$

$2x = 10 \Rightarrow x = 5$

$y = 1$

$\boxed{(5, 1, 0)}$

Point between:

$\boxed{(3, 0, 1)}$

2.1.5 The first of these equations plus the second equals the third:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 2 \\ x & + & 2y & + & z & = & 3 \\ 2x & + & 3y & + & 2z & = & 5 \end{array}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also satisfy the third The equations have infinitely many solutions (the whole line **L**). Find three solutions on **L**.

$x = 0$
$y + z = 2$
$2y + z = 3$
$y = 1,\ z = 1$

$y = 0$
$x + z = 2$
$x + z = 3$
No solution

$z = 0$
$x + y = 2$
$x + 2y = 3$
$x = 1,\ y = 1$

2 points $\boxed{(0,1,1)}$, $\boxed{(1,1,0)}$

3rd point is midpoint between them

$$\boxed{\left(\frac{1}{2}, 1, \frac{1}{2}\right)}$$

Note: There are many other possibilities.

2.1.9 Compute each $A\mathbf{x}$ by dot products of the rows with the column vector:

(a) $\begin{pmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}.$

$(1\ 2\ 4)\cdot(2\ 2\ 3) = 2+4+12 = 18$

$(-2\ 3\ 1)\cdot(2\ 2\ 3) = -4+6+3 = 5$

$(-4\ 1\ 2)\cdot(2\ 2\ 3) = -8+2+6 = 0$

$A\vec{x} = \begin{pmatrix} 18 \\ 5 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$

$(2\ 1\ 0\ 0)\cdot(1\ 1\ 1\ 2) = 2+1 = 3$

$(1\ 2\ 1\ 0)\cdot(1\ 1\ 1\ 2) = 1+2+1 = 4$

$(0\ 1\ 2\ 1)\cdot(1\ 1\ 1\ 2) = 1+2+2 = 5$

$(0\ 0\ 1\ 2)\cdot(1\ 1\ 1\ 2) = 1+4 = 5$

$A\vec{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 5 \end{pmatrix}$

2.1.13 (a) A matrix with m rows and n columns multiplies a vector with ______ n ______ components to produce a vector with ____ m _______ components.

(b) The planes from the m equations $A\mathbf{x} = \mathbf{b}$ are in ______ n ______-dimensional space. The combination of the columns of A is in ______ m ______-dimensional space.

2.1.17 Find the matrix P that multiplies $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to give $\begin{pmatrix} y \\ z \\ x \end{pmatrix}$. Find the matrix Q that multiplies $\begin{pmatrix} y \\ z \\ x \end{pmatrix}$ to bring back $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ x \end{pmatrix}$$

$$\boxed{P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\boxed{Q = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

Note:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As we'd expect, as Q <u>inverts</u> P.

2 Section 2.2 - The Idea of Elimination

2.2.3 What multiple of equations 1 should be *subtracted* from equation 2?

$$\begin{array}{rcrcl} 2x & - & 4y & = & 6 \\ -x & + & 5y & = & 0 \end{array}.$$

After this elimination step, solve the triangular system. If the right side changes to $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, what is the new solution?

$-\frac{1}{2}$ equation 1 should be subtracted from equation 2. This produces

$$2x - 4y = 6$$
$$3y = 3$$

$\Rightarrow y = 1$ $\quad 2x - 4(1) = 6 \Rightarrow 2x = 10 \Rightarrow x = 5$

So, $\boxed{x = 5,\ y = 1}$

If we had $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ we get

$$2x - 4y = -6$$
$$3y = -3$$

$y = -1$

$2x = -10$

$x = -5$

$\boxed{x = -5,\ y = -1}$

2.2.6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case,

$$\begin{aligned} 2x &+ by &= 16 \\ 4x &+ 8y &= g \end{aligned}.$$

We add -2 times the first row to eliminate $4x$ in the second. This will also eliminate $8y$, making it singular, if $b=4$.

$$\begin{aligned} 2x+4y&=16 \\ 4x+8y&=g \end{aligned} \Rightarrow \begin{aligned} 2x+4y&=16 \\ 0y&=g-32 \end{aligned}$$

Only true if $g=32$.

So, $b=4$, $g=32$.

One solution: Pick $x=0$, $y=4$.

Another solution: Pick $y=0$, $x=8$.

So, 2 solutions are $(0,4)$, $(8,0)$.

There are many others.

2.2.11 (Recommended) A system of linear equations can't have exactly two solutions. Why?[1]

(a) If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ are two solutions, what is another solution?

$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\vec{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

If $A\vec{x} = \vec{b}$ and $A\vec{X} = \vec{b}$ then ~~If~~

$A\left(c\vec{x} + (1-c)\vec{X}\right) = \left(c + (1-c)\right)A\vec{x} = \cancel{b}(c+(1-c))\vec{b}$
$= \vec{b}.$

So, any point on the line that goes through $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ is also a solution.

(b) If 25 planes meet at two points, where else do they meet?

~~An~~ At every point on the line through those two points.

[1] You're not begin asked to answer "why" here. Parts (a) and (b) lead you through an explanation as to why.

2.2.12 Reduce this system to upper triangular form by two row operations

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & z & = & 8 \\ 4x & + & 7y & + & 5z & = & 20 \\ & - & 2y & + & 2z & = & 0 \end{array} .$$

Circle the pivots. Solve by back substitution for z, y, x.

Subtract 2x first row from second.

$$\begin{aligned} 2x+3y+z&=8 \\ y+3z&=4 \\ -2y+2z&=0 \end{aligned}$$

~~Subtract~~ Add 2·x second row to third

$$\boxed{\begin{aligned} 2x+3y+z&=8 \\ y+3z&=4 \\ 8z&=8 \end{aligned}}$$

$\Rightarrow \boxed{z=1,\ y=1,\ x=2}$

2.2.19 Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{aligned} x + 4y - 2z &= 1 \\ x + 7y - 6z &= 6 \,. \\ 3y + qz &= t \end{aligned}$$

First elimination step we subtract row 1 from row 2:

$$\begin{aligned} x + 4y - 2z &= 1 \\ 3y - 4z &= 5 \\ 3y + qz &= t \end{aligned}$$

Singular if $\boxed{q = -4.}$

Infinitely many solutions if $\boxed{t = 5.}$

If $q = -4$ and $t = 5$, then if $z = 1$ we get:

$$3y - 4 = 5 \Rightarrow 3y = 9 \Rightarrow y = 3$$

$$x + 7(3) - 6(1) = 6 \Rightarrow x = 12 - 21 = -9.$$

So, $\boxed{x = -9,\ y = 3,\ z = 1}$ is the solution with $z = 1$.