Math 2270 - Assignment 1

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Fall 2012

Section 1.1 - 1,2,13,16,30
Section 1.2 - 1,2,3,27,29
Section 1.3 - 1,2,6,8,13

1 Section 1.1 - Vectors and Linear Combinations

1.1.1 Describe geometrically (line, plane, or all of $\mathbb{R}^3$) all linear combinations of

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(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$.

$$3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \qquad \boxed{\text{Line}}$$

(b) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$,

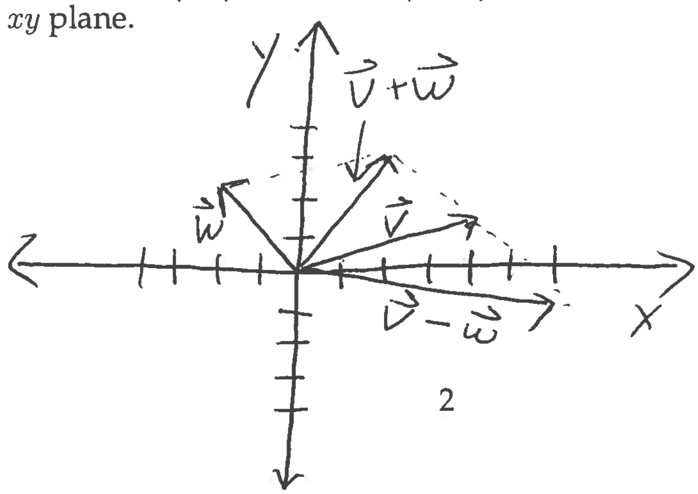
so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ span a $\boxed{\text{plane.}}$

(c) $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

If $a\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

then from row 1 we get $a=1$, and from row 2 we get $b=1$. But, if $a=b=1$ then $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$. So, all of $\boxed{\mathbb{R}^3}$

1.1.2 Draw $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$ and $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ in a single xy plane.



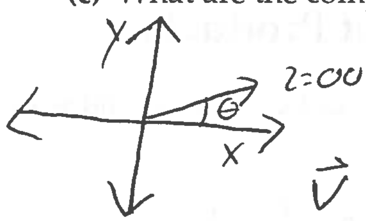
1.1.13 (a) What is the sum **V** of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?

All cancel to produce the zero vector $\vec{0}$.

(b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?

Each one cancels its opposite except the 8:00 vector, which is no longer cancelled by 2:00.

(c) What are the components of that 2:00 vector $\mathbf{v} = (\cos\theta, \sin\theta)$?

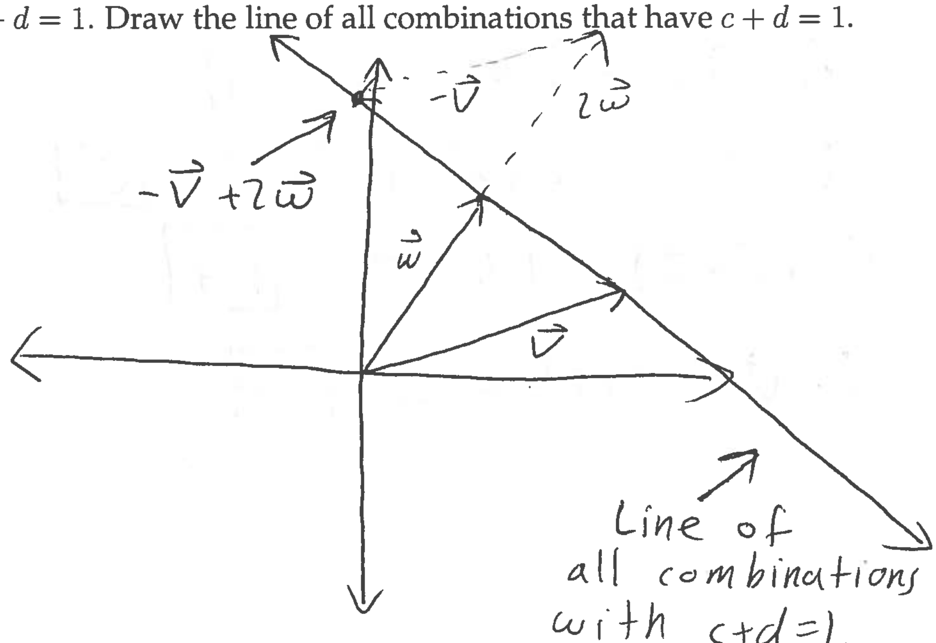


2:00 is $\frac{1}{3}$ of the way up from 3:00 (x-axis) to 12:00 (y-axis).

So, $\theta = \pi/6$

$$\vec{v} = (\cos \pi/6, \sin \pi/6) = \boxed{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)}$$

1.1.16 Mark the point $-\mathbf{v} + 2\mathbf{w}$ and any other combination $c\mathbf{v} + d\mathbf{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.



1.1.30 The linear combinations of $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$ fill the plane unless $\underline{ad = bc}$. Find four vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ with four components each so that their combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w} + f\mathbf{z}$ produce all vectors (b_1, b_2, b_3, b_4) in four-dimensional space.

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{z} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

For example

2 Section 1.2 - Lengths and Dot Products

1.2.1 Calculate the dot products $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$ and $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{u} = \begin{pmatrix} -.6 \\ .8 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}.$$

$$\vec{u} \cdot \vec{v} = -1.8 + 3.2 = \boxed{1.4}$$

$$\vec{u} \cdot \vec{w} = (-.6)(8) + (.8)(6) = \boxed{0}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = 1.4 + 0 = \boxed{1.4}$$

$$\vec{w} \cdot \vec{v} = 3(8) + 4(6) = \boxed{48}$$

1.2.2 Compute the lengths $||\mathbf{u}||$ and $||\mathbf{v}||$ and $||\mathbf{w}||$ of those vectors. Check the Schwarz inequalities $|\mathbf{u}\cdot\mathbf{v}| \le ||\mathbf{u}||||\mathbf{v}||$ and $|\mathbf{v}\cdot\mathbf{w}| \le ||\mathbf{v}||||\mathbf{w}||$.

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$$||u|| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{24}} = \sqrt{1} = \boxed{1}$$

$$||v|| = \sqrt{3^2 + 4^2} = \sqrt{25} = \boxed{5}$$

$$||w|| = \sqrt{8^2 + 6^2} = \sqrt{100} = \boxed{10}$$

$$|\vec{u}\cdot\vec{v}| = \boxed{1.4} < (1)(5) = \boxed{5} \checkmark$$

$$|\vec{w}\cdot\vec{w}| = \boxed{48} < 5(10) = \boxed{50} \checkmark$$

1.2.3 Find unit vectors in the directions of **v** and **w** in Problem 1, and the cosine of the angle θ. Choose vectors **a**, **b**, **c** that make 0°, 90°, and 180° angles with **w**.

$$\vec{v} = \langle 3, 4 \rangle \qquad \hat{v} = \frac{1}{5}\langle 3, 4 \rangle = \begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$$

$$\vec{w} = \langle 8, 6 \rangle \qquad \hat{w} = \frac{1}{10}\langle 8, 6 \rangle = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{||\vec{v}||\,||\vec{w}||} = \frac{48}{50} = \boxed{\frac{24}{25}}$$

$$\vec{a} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}, \qquad \vec{c} = \begin{pmatrix} -\frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$$

1.2.27 (*Recommended*) If $||\mathbf{v}|| = 5$ and $||\mathbf{w}|| = 3$, what are the smallest and largest values of $||\mathbf{v} - \mathbf{w}||$? What are the smallest and largest values of $\mathbf{v} \cdot \mathbf{w}$?

Largest if $\theta = 180^\circ$ and $\vec{v}, \vec{w}$ in opposite directions. Smallest if $\theta = 0^\circ$ and $\vec{v}, \vec{w}$ in same direction

Smallest $||\vec{v} - \vec{w}|| = 2$.
Largest $||\vec{v} - \vec{w}|| = 8$.
Smallest $\vec{v} \cdot \vec{w} = \cancel{-15}\ -15$
Largest $\vec{v} \cdot w = 15$.

1.2.29 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$. Challenge question: Explain why $\mathbf{v} \cdot \mathbf{w}/||\mathbf{v}||||\mathbf{w}||$ is always $-\frac{1}{2}$.

$z = -x - y$.

$(x, y, -x-y) \cdot (-x-y, x, y) = -x^2 - xy + xy - xy - y^2$

$= -x^2 - xy - y^2$

$||\vec{v}|| = \sqrt{x^2 + y^2 + (-x-y)^2} = \sqrt{2x^2 + 2y^2 + 2xy}$

$||\vec{w}|| = \sqrt{(-x-y)^2 + x^2 + y^2} = \sqrt{2x^2 + 2y^2 + 2xy}$

$\cos\theta = \dfrac{-x^2 - xy - y^2}{2x^2 + 2xy + y^2} = \boxed{-\frac{1}{2}}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \boxed{120^\circ}$

Our vectors will always be 2 parts of a Y shape Y whose third component is (y, z, x).

3 Section 1.3 - Matrices

1.3.1 Find the linear combinations $2\mathbf{s}_1 + 3\mathbf{s}_2 + 4\mathbf{s}_3 = \mathbf{b}$. Then write $\mathbf{b}$ as a matrix-vector multiplication $S\mathbf{x}$. Compute the dot products (row of S)$\cdot\mathbf{x}$:

$$\mathbf{s}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{s}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ go into the columns of } S.$$

$$2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}$$

$$(1\,0\,0)\cdot(2\,3\,4) = 2$$

$$(1\,1\,0)\cdot(2\,3\,4) = 5$$

$$(1\,1\,1)\cdot(2\,3\,4) = 9$$

1.3.2 Solve these equations $S\mathbf{y} = \mathbf{b}$ with $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ in the columns of S:

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$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}.$$

The sum of the first n odd numbers is $\underline{\quad n^2 \quad}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1+y_2 \\ y_1+y_2+y_3 \end{pmatrix}$$

If $\begin{pmatrix} y_1 \\ y_1+y_2 \\ y_1+y_2+y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\boxed{\begin{matrix} y_1 = 1 \\ y_2 = 0 \\ y_3 = 0 \end{matrix}}$

If $\begin{pmatrix} y_1 \\ y_1+y_2 \\ y_1+y_2+y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}$ $\boxed{\begin{matrix} y_1 = 1 \\ y_2 = 3 \\ y_3 = 5 \end{matrix}}$

1.3.6 Which values of c give dependent columns (combination equals zero)?

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{pmatrix}.$$

$$\begin{aligned} a + 3b &= 5 \\ a + 2b &= 4 \end{aligned} \quad \Rightarrow b = 1, \ a = 2$$

$$2 + 1 = c \Rightarrow c = 3.$$

For first matrix $\begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{pmatrix}$ we have $\boxed{c=3}$

$$\begin{aligned} a + 0 &= c \\ a + b &= 0 \qquad a = -1 \\ b &= 1 \qquad c = -1. \end{aligned}$$

For second matrix $\begin{pmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ we have $\boxed{c=-1}$

$$\begin{aligned} 2a + b &= 5 \\ 3a + 3b &= 6 \\ b &= 5 - 2a \end{aligned} \qquad \begin{aligned} 3c - c &= c \\ 2c &= c \\ \Rightarrow c &= 0. \end{aligned}$$

For third matrix

$$\begin{pmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{pmatrix} \quad \boxed{c = 0}.$$

$$3a + 15 - 6a = 6$$

$$-3a = -9$$

$$a = +3 \quad b = -1$$

1.3.8 Moving to a 4 by 4 difference equation $A\mathbf{x} = \mathbf{b}$, find the four components x_1, x_2, x_3, x_4. Then write this solution as $\mathbf{x} = S\mathbf{b}$ to find the inverse matrix $S = A^{-1}$:

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \mathbf{b}.$$

$$\begin{aligned} x_1 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \implies \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned}$$

$$\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$A^{-1} = S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

1.3.13 The very last words of worked example 1.3B say that the 5 by 5 centered difference matrix *is not* invertible. Write down the 5 equations $C\mathbf{x} = \mathbf{b}$. Find the combination of left sides that gives zero. What combination of b_1, b_2, b_3, b_4, b_5 must be zero? (The 5 columns lie on a "4-dimensional hyperplane" in 5-dimensional space.)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$$

$$x_2 = b_1$$
$$x_3 - x_1 = b_2$$
$$x_4 - x_2 = b_3$$
$$x_5 - x_3 = b_4$$
$$-x_4 = b_5$$

$\vec{b} = \vec{0}$ when

$x_2 = 0, \ x_4 = 0$

$x_1 = x_3 = x_5$.

We note that $b_1 + b_3 + b_5 = 0$.
So, a solution is only possible if $b_1 + b_3 + b_5 = 0$.