Math 2270 - Assignment 14

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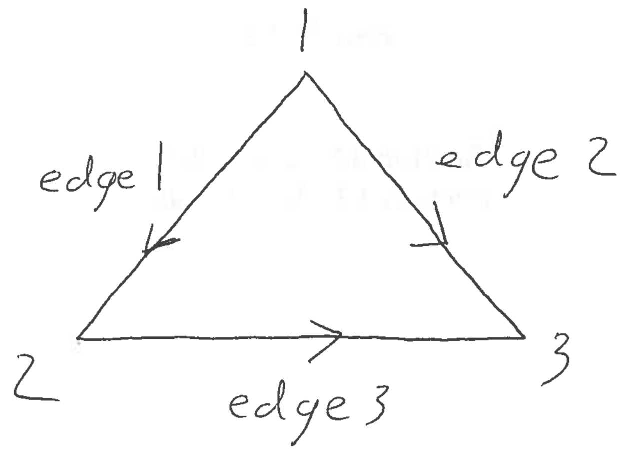
Section 8.2 - 1, 2, 3, 4, 5
Section 8.3 - 1, 2, 3, 9, 10

8.2 - Graphs and Networks

8.2.1 - Write down the 3×3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (x_1, x_2, x_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

The triangle graph looks like:



Incidence Matrix
$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

the nullspace is span $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is not in the row space because it's not perpendicular to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

8.2.2 - Write down A^T for the triangle graph. Find a vector **y** in its nullspace. The components of **y** are current on the edges - how much current is going around the triangle?

$$A^T = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\vec{y} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is in its nullspace.

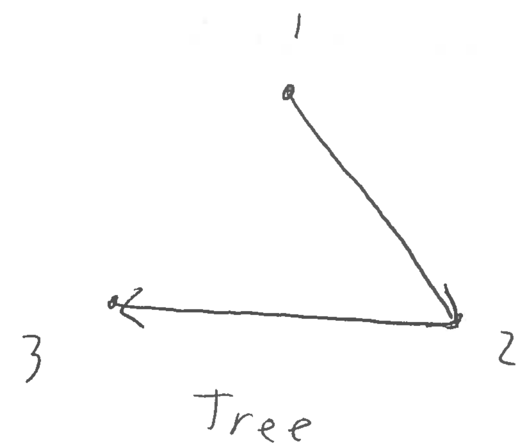
A current of 1 unit is going around the triangle

8.2.3 - Eliminate x_1 and x_2 from the third equation to find the echelon matrix U. What tree corresponds to the two nonzero rows of U?

$$\begin{aligned} -x_1 &+ x_2 &= b_1 \\ -x_1 &+ x_3 &= b_2 \\ -x_2 &+ x_3 &= b_3 \end{aligned} .$$

$$\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{\text{subtract row 1 from row 2}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{subtract row 2 from row 3}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Tree

8.2.4 - Choose a vector (b_1, b_2, b_3) for which $A\mathbf{x} = \mathbf{b}$ can be solved, and another vector $\mathbf{b}$ that allows no solution. How are those $\mathbf{b}$'s related to $\mathbf{y} = (1, -1, 1)$?

$$-x_1 \quad + x_3 = b_1$$
$$-x_2 \quad + x_3 = b_2 - b_1$$
$$0 = b_3 - b_2 + b_1$$

$\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ works $\quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ does not.

$\vec{b}$ is a solution if and only if $\vec{b} \cdot \vec{y} = 0$.

8.2.5 - Choose a vector (f_1, f_2, f_3) for which $A^T\mathbf{y} = \mathbf{f}$ can be solved, and a vector $\mathbf{f}$ that allows no solution. How are those $\mathbf{f}$'s related to $\mathbf{x} = (1,1,1)$? The equation $A^T\mathbf{y} = \mathbf{f}$ is Kirchoff's current law.

$$A^T\vec{y} = \vec{f}$$

$$A^T\vec{y} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -y_1 - y_2 \\ y_1 - y_3 \\ y_2 + y_3 \end{pmatrix}$$

Pick $\vec{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \vec{f} = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$

$\vec{f} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ does not work.

$\vec{f}$ works if and only if $\vec{f} \cdot \vec{x} = 0$.

Markov Matrices, Population, and Economics

8.3.1 - Find the eigenvalues of this Markov matrix (their sum is the trace):

$$A = \begin{pmatrix} .90 & .15 \\ .10 & .85 \end{pmatrix}$$

$\lambda = 1$ is an eigenvalue necessarily.

$.90 + .85 - 1 = .75$ must be the other eigenvalue.

8.3.2 - Diagonalize the Markov matrix in Problem 1 to $A = S\Lambda S^{-1}$ by finding its other eigenvector:

$$A = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} 1 & \\ & .75 \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}.$$

What is the limit of $A^k = S\Lambda^k S^{-1}$ when $\Lambda^k = \begin{pmatrix} 1 & 0 \\ 0 & .75^k \end{pmatrix}$ approaches $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$?

$\lambda = 1$

$$\begin{pmatrix} -.10 & .15 \\ .10 & -.15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\lambda = .75$

$$\begin{pmatrix} .15 & .15 \\ .10 & .10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} \qquad S^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & .75 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}}$$

$$\lim_{k\to\infty} A^k = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{3}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{pmatrix}}$$

8.3.3 - What are the eigenvalues and steady state eigenvectors for these Markov matrices?

$$A = \begin{pmatrix} 1 & .2 \\ 0 & .8 \end{pmatrix} \qquad A = \begin{pmatrix} .2 & 1 \\ .8 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

$A = \begin{pmatrix} 1 & .2 \\ 0 & .8 \end{pmatrix}$ Eigenvalues $1, .8$

$\begin{pmatrix} 0 & .2 \\ 0 & -.2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is steady state eigenvector.

$A = \begin{pmatrix} .2 & 1 \\ .8 & 0 \end{pmatrix}$ Eigenvalues $1, -.8$

$\begin{pmatrix} -.8 & 1 \\ .8 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} \frac{5}{4} \\ 1 \end{pmatrix}$ $\begin{pmatrix} \frac{5}{9} \\ \frac{4}{9} \end{pmatrix}$ is steady state eigenvector.

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} = \left(\frac{1}{2}-\lambda\right)^3 + \frac{1}{64} + \frac{1}{64} - \frac{3}{16}\left(\frac{1}{2}-\lambda\right)$$

$$= \frac{1}{8} - \frac{3}{4}\lambda + \frac{3}{2}\lambda^2 - \lambda^3 + \frac{1}{32} - \frac{3}{32} + \frac{3}{16}\lambda$$

$$= -\lambda^3 + \frac{3}{2}\lambda^2 - \frac{9}{16}\lambda + \frac{1}{16} = (1-\lambda)\left(\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16}\right)$$

$$= (1-\lambda)\left(\lambda - \frac{1}{4}\right)^2$$

Eigenvalues $\lambda = 1, \frac{1}{4}, \frac{1}{4}$

$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \leftarrow$ steady state eigenvector.

8.3.9 - Prove that the square of a Markov matrix is also a Markov matrix.

If A is Markov then if the sum of the terms of $\vec{u}$ is 1 then $A\vec{u}$ sums to 1 as well.

$$A^2 = A\begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{pmatrix} = \begin{pmatrix} A\vec{a}_1 & A\vec{a}_2 & \cdots & A\vec{a}_n \end{pmatrix}$$

Each $A\vec{a}_i$ ~~sumt~~ sums to 1, and so A^2 is Markov.

8.3.10 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Markov matrix, its eigenvalues are 1 and $\underline{a+d-1}$. The steady state eigenvector is $\mathbf{x}_1 =$ ____________.

$$\begin{pmatrix} \dfrac{1-d}{2-a-d} \\[2em] \dfrac{1-a}{2-a-d} \end{pmatrix}$$