Key

Math 2270 - Assignment 13

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Section 7.1 - 1, 3, 4, 10, 16
Section 7.2 - 5, 14, 15, 17, 26
Section 7.3 - 1, 5, 6, 7, 9

7.1 - The Idea of a Linear Transformation

7.1.1 - A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ by choosing $\mathbf{w} =$ $\underline{\quad \vec{0} \quad}$. Prove it also from $T(c\mathbf{v}) = cT(\mathbf{v})$ by choosing $c =$ $\underline{\quad 0 \quad}$.

$$T(\vec{v}) = T(\vec{v} + \vec{0}) = T(\vec{v}) + T(\vec{0})$$

$$\Rightarrow T(\vec{0}) = \vec{0}$$

$$T(\vec{0}) = T(0\,\vec{v}) = 0\,T(\vec{v}) = \vec{0}$$

7.1.3 - Which of these transformations are not linear? The input is $\mathbf{v} = (v_1, v_2)$:

(a) $T(\mathbf{v}) = (v_2, v_1)$ Linear

(b) $T(\mathbf{v}) = (v_1, v_1)$ Linear

(c) $T(\mathbf{v}) = (0, v_1)$ Linear

(d) $T(\mathbf{v}) = (0, 1)$ Not linear

(e) $T(\mathbf{v}) = v_1 - v_2$ Linear

(f) $T(\mathbf{v}) = v_1 v_2$ Not linear

7.1.4 - If S and T are linear transformations, is $S(T(\mathbf{v}))$ linear or quadratic?

(a) (Special case) If $S(\mathbf{v}) = \mathbf{v}$ and $T(\mathbf{v}) = \mathbf{v}$, then $S(T(\mathbf{v})) = \mathbf{v}$ or $\mathbf{v}^2$?

(b) (General case) $S(\mathbf{w}_1 + \mathbf{w}_2) = S(\mathbf{w}_1) + S(\mathbf{w}_2)$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ combine into

$$S(T(\mathbf{v}_1 + \mathbf{v}_2)) = S(\underline{T(\vec{v}_1) + T(\vec{v}_2)}) = \underline{S(T(\vec{v}_1))} + \underline{S(T(\vec{v}_2))}.$$

a) $S(T(\vec{v})) = S(\vec{v}) = \vec{v}$.

b) $S(T(\vec{v}_1 + \vec{v}_2)) = S(T(\vec{v}_1)) + S(T(\vec{v}_2))$

7.1.10 - A linear transformation from **V** to **W** has an *inverse* from **W** to **V** when the range is all of **W** and the kernel contains only $\mathbf{v} = \mathbf{0}$. Then $T(\mathbf{v}) = \mathbf{w}$ has one solutions **v** for each **w** in **W**. Why are these T's not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\quad\quad \mathbf{W} = \mathbb{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\quad\quad \mathbf{W} = \mathbb{R}^3$

(c) $T(v_1, v_2) = v_1$ $\quad\quad \mathbf{W} = \mathbb{R}^1$

a) Output is not all of $\mathbb{R}^2$. For example, $(1,2)$ is not in the output.

b) Output not all of $\mathbb{R}^3$. For example, $(1, 1, 3)$ is not in the output.

c) kernel is not $\vec{0}$. For example, $T(0, 5) = 0$.

7.1.16 - Suppose T transposes every matrix M. Try to find a matrix A which gives $AM = M^T$ for every M. Show that no matrix A will do it. *To professors*: Is this a linear transformation that doesn't come from a matrix?

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{11} \\ 0 & a_{21} \end{pmatrix}$$

$$\neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{for any } a_{11}, a_{12}, a_{21}, a_{22}$$

Nope. $M \to M^T$ is a map from $\mathbb{R}^4$ to $\mathbb{R}^4$. It would be represented by the 4×4 matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

7.2 - The Matrix of a Linear Transformation

7.2.5 - With bases $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, suppose $T(\mathbf{v}_1) = \mathbf{w}_2$ and $T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{w}_1 + \mathbf{w}_3$. T is a linear transformation. Find the matrix A and multiply by the vector $(1, 1, 1)$. What is the output from T when the input is $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$?

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$T(\vec{v}_1 + \vec{v}_2 + \vec{v}_3) = 2\vec{w}_1 + \vec{w}_2 + 2\vec{w}_3$$

7.2.14 -

(a) What matrix transforms $(1, 0)$ into $(2, 5)$ and $(0, 1)$ to $(1, 3)$?

(b) What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

(c) Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

a) $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$

c) Because $(2,6) = 2(1,3)$.
So, if $A(1,3) = (0,1)$
$A(2,6) = (0,2)$, not $(1,0)$.

7.2.15 -

(a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?

(b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?

(c) What condition on a,b,c,d will make part (b) impossible?

a) $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$

b) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

c) If $ad-bc=0$.

7.2.17 - If you keep the same basis vectors but put them in a different order, the change of basis matrix M is a permutation matrix. If you keep the basis vectors in order but change their lengths, M is a diagonal matrix.

7.2.26 - Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are **eigenvectors** for T. This means $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$. What is the matrix for T when the input and output bases are the $\mathbf{v}$'s?

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

7.3 - Diagonalization and the Pseudoinverse

7.3.1 -

(a) Compute A^TA and its eigenvalues and unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Find σ_1.

Rank one matrix $$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

(b) Compute AA^T and its eigenvalues and unit eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

(c) Verify that $A\mathbf{v}_1 = \sigma_1\mathbf{u}_1$. Put numbers into the SVD:

$$A = U\Sigma V^T$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \\ & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

a) $A^TA = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix}$

$\begin{vmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{vmatrix} = (10-\lambda)(40-\lambda) - 400$
$= \lambda^2 - 50\lambda = (\lambda-50)\lambda$

$\lambda = 50$ $\begin{pmatrix} -40 & 20 \\ 20 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$\lambda = 0$ $\begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\vec{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$\sigma_1 = \sqrt{50}$

Here is some extra space for problem 7.3.1 if you need it.

b)

$$AA^T = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix}$$

$$\begin{vmatrix} 5-\lambda & 15 \\ 15 & 45-\lambda \end{vmatrix} = (5-\lambda)(45-\lambda) - 15^2$$
$$= \lambda^2 - 50\lambda = (\lambda - 50)\lambda$$

$\lambda = 50$

$$\begin{pmatrix} -45 & 15 \\ 15 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \vec{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

$\lambda = 0$

$$\begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 3\sqrt{5} \end{pmatrix} = \sqrt{50}\begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

7.3.5 - Compute A^TA and is eigenvalues and unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. What are the singular values σ_1 and σ_2 for this matrix A?

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} \qquad A^TA = \begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 8 \\ 8 & 10 \end{pmatrix}$$

$$\begin{vmatrix} 10-\lambda & 8 \\ 8 & 10-\lambda \end{vmatrix} = (10-\lambda)^2 - 64 = \lambda^2 - 20\lambda + 36 = (\lambda - 18)(\lambda - 2)$$

$$\lambda = 18 \quad \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\sigma_1 = \sqrt{18} = 3\sqrt{2} \qquad \sigma_2 = \sqrt{2}$$

7.3.6 - AA^T has the same eigenvalues σ_1^2 and σ_2^2 as A^TA. Find unit eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Put numbers into the SVD:

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

$$AA^T = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda = 18 \quad \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = 2 \quad \begin{pmatrix} 16 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

7.3.7 In Problem 6, multiply columns times rows to show that $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$. Prove from $A = U\Sigma V^T$ that every matrix of rank r is the sum of r matrices of rank one.

$$\vec{u}_1 \vec{v}_1^{\,T} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$\vec{u}_2 \vec{v}_2^{\,T} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = 3\sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} + \sqrt{2} \begin{pmatrix} 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$U \Sigma V^T = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^{\,T}$$

7.3.9 The pseudoinverse of this A is the same as $\underline{\quad A^{-1} \quad}$ because
<u>A is invertible.</u>