Math 2270 - Assignment 13

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Section 7.1 - 1, 3, 4, 10, 16
Section 7.2 - 5, 14, 15, 17, 26
Section 7.3 - 1, 5, 6, 7, 9

7.1 - The Idea of a Linear Transformation

7.1.1 - A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ by choosing $\mathbf{w} =$ ____________. Prove it also from $T(c\mathbf{v}) = cT(\mathbf{v})$ by choosing $c =$ ____________.

7.1.3 - Which of these transformations are not linear? The input is $\mathbf{v} = (v_1, v_2)$:

(a) $T(\mathbf{v}) = (v_2, v_1)$

(b) $T(\mathbf{v}) = (v_1, v_1)$

(c) $T(\mathbf{v}) = (0, v_1)$

(d) $T(\mathbf{v}) = (0, 1)$

(e) $T(\mathbf{v}) = v_1 - v_2$

(f) $T(\mathbf{v}) = v_1 v_2$

7.1.4 - If S and T are linear transformations, is $S(T(\mathbf{v}))$ linear or quadratic?

(a) (Special case) If $S(\mathbf{v}) = \mathbf{v}$ and $T(\mathbf{v}) = \mathbf{v}$, then $S(T(\mathbf{v})) = \mathbf{v}$ or $\mathbf{v}^2$?

(b) (General case) $S(\mathbf{w}_1 + \mathbf{w}_2) = S(\mathbf{w}_1) + S(\mathbf{w}_2)$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ combine into

$$S(T(\mathbf{v}_1 + \mathbf{v}_2)) = S(\underline{\hspace{3cm}}) = \underline{\hspace{3cm}} + \underline{\hspace{3cm}}.$$

7.1.10 - A linear transformation from **V** to **W** has an *inverse* from **W** to **V** when the range is all of **W** and the kernel contains only $\mathbf{v} = \mathbf{0}$. Then $T(\mathbf{v}) = \mathbf{w}$ has one solutions $\mathbf{v}$ for each $\mathbf{w}$ in **W**. Why are these T's not invertible?

(a) $T(v_1, v_2) = (v_2, v_2)$ $\qquad$ $\mathbf{W} = \mathbb{R}^2$

(b) $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\qquad$ $\mathbf{W} = \mathbb{R}^3$

(c) $T(v_1, v_2) = v_1$ $\qquad$ $\mathbf{W} = \mathbb{R}^1$

7.1.16 - Suppose T transposes every matrix M. Try to find a matrix A which gives $AM = M^T$ for every M. Show that no matrix A will do it. *To professors*: Is this a linear transformation that doesn't come from a matrix?

7.2 - The Matrix of a Linear Transformation

7.2.5 - With bases $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, suppose $T(\mathbf{v}_1) = \mathbf{w}_2$ and $T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{w}_1 + \mathbf{w}_3$. T is a linear transformation. Find the matrix A and multiply by the vector $(1, 1, 1)$. What is the output from T when the input is $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$?

7.2.14 -

(a) What matrix transforms $(1, 0)$ into $(2, 5)$ and $(0, 1)$ to $(1, 3)$?

(b) What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

(c) Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?

7.2.15 -

(a) What matrix M transforms $(1,0)$ and $(0,1)$ to (r,t) and (s,u)?

(b) What matrix N transforms (a,c) and (b,d) to $(1,0)$ and $(0,1)$?

(c) What condition on a, b, c, d will make part (b) impossible?

7.2.17 - If you keep the same basis vectors but put them in a different order, the change of basis matrix M is a _________________________ matrix. If you keep the basis vectors in order but change their lengths, M is a _________________________ matrix.

7.2.26 - Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are **eigenvectors** for T. This means $T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$ for $i = 1, 2, 3$. What is the matrix for T when the input and output bases are the **v**'s?

7.3 - Diagonalization and the Pseudoinverse

7.3.1 -

(a) Compute $A^T A$ and its eigenvalues and unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Find σ_1.

Rank one matrix $$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

(b) Compute AA^T and its eigenvalues and unit eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

(c) Verify that $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$. Put numbers into the SVD:

$$A = U\Sigma V^T$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \\ & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

Here is some extra space for problem 7.3.1 if you need it.

7.3.5 - Compute $A^T A$ and is eigenvalues and unit eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. What are the singular values σ_1 and σ_2 for this matrix A?

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}$$

7.3.6 - AA^T has the same eigenvalues σ_1^2 and σ_2^2 as A^TA. Find unit eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Put numbers into the SVD:

$$A = \begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

7.3.7 In Problem 6, multiply columns times rows to show that $A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$. Prove from $A = U \Sigma V^T$ that every matrix of rank r is the sum of r matrices of rank one.

7.3.9 The pseudoinverse of this A is the same as ____________ because ____________.