Math 2270 - Assignment 12

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Section 6.5 - 2, 3, 7, 11, 16
Section 6.6 - 3, 9, 10, 12, 13
Section 6.7 - 1, 4, 6, 7, 9

6.5 - Positive Definite Matrices

6.5.2 - Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compue the λ's. Find an $\mathbf{x}$ so that $\mathbf{x}^T A_1 \mathbf{x} < 0$, so A_1 fails the test.

$$A_1 = \begin{pmatrix} 5 & 6 \\ 6 & 7 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 10 \\ 10 & 100 \end{pmatrix}$$
$$A_4 = \begin{pmatrix} 1 & 10 \\ 10 & 101 \end{pmatrix}.$$

A_1 $\quad 5 > 0, \quad 5 \times 7 - 6^2 = -1.$

So, not positive definite.

A_2 $\quad -1 < 0,$ so not positive definite.

A_3 $\quad 1 > 0, \quad 1 \times 100 - 10 \times 10 = 0.$

So, not positive definite.

A_4 $\quad 1 > 0, \quad 1 \times 101 - 10 \times 10 = 1.$

So, positive definite.

$$(x_1\ x_2)\begin{pmatrix} 5 & 6 \\ 6 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (5x_1 + 6x_2 \quad 6x_1 + 7x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= 5x_1^2 + 12x_1x_2 + 7x_2^2$$
$$= (5x_1 + 7x_2)(x_1 + x_2)$$

If $x_1 = 1$, $x_2 = -1$ then this product is 0.

So, $\vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ works.

6.5.3 - For which numbers b and c are these matrices positive definite?

$$A = \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 4 \\ 4 & c \end{pmatrix} \qquad A = \begin{pmatrix} c & b \\ b & c \end{pmatrix}$$

$\begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix}$ Positive definite if

$9 - b^2 > 0 \Rightarrow 9 > b^2$

So, $-3 < b < 3$.

$\begin{pmatrix} 2 & 4 \\ 4 & c \end{pmatrix}$ Positive definite if

$2c - 16 > 0$

$\Rightarrow c > 8$.

$\begin{pmatrix} c & b \\ b & c \end{pmatrix}$ Positive definite if

$c > 0$ and $c^2 - b^2 > 0$

$\Rightarrow c^2 > b^2$

So, $c > |b|$

6.5.7 - Test to see if $R^T R$ is positive definite in each case:

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and}$$
$$R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad R^T R = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 13 \end{pmatrix}$$

Positive definite.

Note: In general $R^T R$ is positive definite if and only if R has independent columns

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$$

The columns are linearly independent, so $R^T R$ is positive definite.

$$R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

The columns are not linearly independent, so $R^T R$ is not positive definite.

6.5.11 - Compute the three upper left determinants of A to establish the positive definiteness. Verify that their ratios give the second and third pivots.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$$

$2 \qquad \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 10 - 4 = 6$

$$\begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{vmatrix} = 2 \begin{vmatrix} 5 & 3 \\ 3 & 8 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix}$$

$$= 2(40-9) - 2(16) = 2(15) = 30$$

So, A is positive definite

$$\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 8 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

$2, \quad 3 = \frac{6}{2}, \quad 5 = \frac{30}{6}$

So, the ratios give the second and third pivots

6.5.16 - A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $\mathbf{x}^T A\mathbf{x} > 0$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ is not positive when } (x_1, x_2, x_3) = (3, -19, 7)$$

$$4x_1 + x_3 = 1$$
$$x_1 + 5x_3 = 2$$

$$-19x_3 = -7$$
$$x_3 = \frac{7}{19}$$

$$4x_1 = 1 - \frac{7}{19} = \frac{12}{19} \Rightarrow x_1 = \frac{3}{19}$$

$$x_2 = -1$$

$$\vec{x} = \begin{pmatrix} 3 \\ -19 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -19 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -19 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 17 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -19 \\ 7 \end{pmatrix} = -323$$

6.6 - Similar Matrices

6.6.3 - Show that A and B are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6.6.9 - By direct multiplication find A^2 and A^3 and A^5 when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Guess the form of A^k. Set $k = 0$ to find A^0 and $k = -1$ to find A^{-1}.

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

6.6.10 - By direct multiplication, find J^2 and J^3 when

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Guess the form of J^k. Set $k = 0$ to find J^0. Set $k = -1$ to find J^{-1}.

$$J^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$

$$J^k = \begin{pmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{pmatrix}$$

$$J^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$J^{-1} = \begin{pmatrix} \frac{1}{\lambda} & -\frac{1}{\lambda^2} \\ 0 & \frac{1}{\lambda} \end{pmatrix}$$

6.6.12 - These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: J *is not similar* to K.

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \qquad JM = \begin{pmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Mk = \begin{pmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{pmatrix}$$

$Mk = JM$

So therefore

$m_{11} = m_{22} = 0$

$m_{21} = 0$

$m_{31} = m_{42} = 0$

$m_{41} = 0$

So, the first column of M is all 0s, and therefore $\det(M) = 0$

6.6.13 Based on Problem 12, what are the five Jordan forms when $\lambda = 0, 0, 0, 0$?

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6.7 - Singular Value Decomposition (SVD)

6.7.1 - Find the eigenvalues and unit eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of A^TA. Then find $\mathbf{u}_1 = A\mathbf{v}_1/\sigma_1$:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \text{ and } A^TA = \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix} \text{ and } AA^T = \begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix}.$$

Verify that $\mathbf{u}_1$ is a unit eigenvectors of AA^T. Complete the matrices U, Σ, V.

$$\text{SVD} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

$$\begin{vmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{vmatrix} = \lambda^2 - 50\lambda = (\lambda-50)\lambda \qquad \lambda = 50, 0$$

$$\lambda = 50 \quad \begin{pmatrix} -40 & 20 \\ 20 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$\lambda = 0 \quad \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \frac{1}{5\sqrt{10}} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{5\sqrt{10}} \begin{pmatrix} 5 \\ 15 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{50}{\sqrt{10}} \\ \frac{150}{\sqrt{10}} \end{pmatrix} = 50 \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} = 50\vec{u}_1$$

So, $\vec{u}_1$ is an eigenvector of AA^T.

$$\begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}$$

The SVD is

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

6.7.4 - Find the eigenvalues and unit eigenvectors of A^TA and AA^T. Keep each $A\mathbf{v} = \sigma\mathbf{u}$:

Fibonacci Matrix $\qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Construct the singular value decomposition and verify that A equals $U\Sigma V^T$.

$$A^T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A^TA = AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1 \quad \text{Roots } \lambda = \frac{3 \pm \sqrt{5}}{2}$$

$$\lambda = \frac{3+\sqrt{5}}{2}$$

$$\begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} \frac{2}{5-\sqrt{5}} \\ \frac{-1+\sqrt{5}}{5-\sqrt{5}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} \frac{2}{5+\sqrt{5}} \\ \frac{-1-\sqrt{5}}{5+\sqrt{5}} \end{pmatrix}$$

$$\sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{2}$$

$$\sigma_2 = \sqrt{\frac{3-\sqrt{5}}{2}} = \frac{\sqrt{5}-1}{2}$$

$$A\vec{v}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5-\sqrt{5}} \\ \frac{-1+\sqrt{5}}{5-\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{5-\sqrt{5}} \\ \frac{2}{5-\sqrt{5}} \end{pmatrix}$$

$$\vec{u}_1 = \frac{2}{1+\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{5-\sqrt{5}} \\ \frac{2}{5-\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{2}{5-\sqrt{5}} \\ \frac{-1+\sqrt{5}}{5-\sqrt{5}} \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5+\sqrt{5}} \\ \frac{-1-\sqrt{5}}{5+\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{5+\sqrt{5}} \\ \frac{2}{5+\sqrt{5}} \end{pmatrix}$$

$$\vec{u}_2 = \frac{2}{\sqrt{5}-1} \begin{pmatrix} \frac{1-\sqrt{5}}{5+\sqrt{5}} \\ \frac{2}{5+\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{-2}{5+\sqrt{5}} \\ \frac{1+\sqrt{5}}{5+\sqrt{5}} \end{pmatrix}$$

So,

$A = U \Sigma V^T$ is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5-\sqrt{5}} & \frac{-2}{5+\sqrt{5}} \\ \frac{-1+\sqrt{5}}{5-\sqrt{5}} & \frac{1+\sqrt{5}}{5+\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{\sqrt{5}-1}{2} \end{pmatrix} \begin{pmatrix} \frac{2}{5-\sqrt{5}} & \frac{-1+\sqrt{5}}{5-\sqrt{5}} \\ \frac{2}{5+\sqrt{5}} & \frac{-1-\sqrt{5}}{5+\sqrt{5}} \end{pmatrix}$$

6.7.6 - Compute $A^T A$ and AA^T and their eigenvalues and unit eigenvectors for this A:

Rectangular Matrix $\qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$

Multiply the matrices $U\Sigma V^T$ to recover A. Σ has the same shape as A.

$$A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(1-\lambda)\right] - 2(1-\lambda)$$
$$= (1-\lambda)\left((2-\lambda)(1-\lambda) - 2\right)$$
$$= (1-\lambda)(\lambda^2 - 3\lambda) = (1-\lambda)(\lambda-3)\lambda$$

$$\lambda = 0, 1, 3$$

$\lambda = 0$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda = 3$

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \vec{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$A\vec{v}_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \Rightarrow \vec{u}_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A\vec{v}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

As $\lambda = 0$ corresponds with $\vec{v}_3$, $\vec{u}_3$ does not enter the picture

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\lambda_1 = 3 \quad \sigma_1 = \sqrt{3}$

$\lambda_2 = 1 \quad \sigma_2 = 1$

$\lambda_3 = 0 \quad \sigma_3$ does not exist. Or, at least, not in this formulation.

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

So,

$$A = U \Sigma V^T$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

6.7.7 - What is the closest rank-one approximation to that 3 by 2 matrix?

Typo, should be 2×3.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\vec{u}_1 \sigma_1 \vec{v}_1^{T} = \sqrt{3} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}$$

6.7.9 - Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal bases for $\mathbb{R}^n$. Construct the matrix A that transforms each $\mathbf{v}_j$ into $\mathbf{u}_j$ to give $A\mathbf{v}_1 = \mathbf{u}_1, \ldots, A\mathbf{v}_n = \mathbf{u}_n$.

$$V = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} \qquad U = \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix}$$

$$V^T V = I$$

$$\Rightarrow U V^T V = U$$

$$\Rightarrow U V^T \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix} = \begin{pmatrix} U V^T \vec{v}_1 & \cdots & U V^T \vec{v}_n \end{pmatrix}$$

$$= \begin{pmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{pmatrix}.$$

So, $\boxed{A = U V^T}$