Math 2270 - Assignment 12

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Section 6.5 - 2, 3, 7, 11, 16
Section 6.6 - 3, 9, 10, 12, 13
Section 6.7 - 1, 4, 6, 7, 9

6.5 - Positive Definite Matrices

6.5.2 - Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compue the λ's. Find an **x** so that $\mathbf{x}^T A_1 \mathbf{x} < 0$, so A_1 fails the test.

$$A_1 = \begin{pmatrix} 5 & 6 \\ 6 & 7 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix} \qquad A_3 = \begin{pmatrix} 1 & 10 \\ 10 & 100 \end{pmatrix}$$
$$A_4 = \begin{pmatrix} 1 & 10 \\ 10 & 101 \end{pmatrix}.$$

6.5.3 - For which numbers b and c are these matrices positive definite?

$$A = \begin{pmatrix} 1 & b \\ b & 9 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 4 \\ 4 & c \end{pmatrix} \qquad A = \begin{pmatrix} c & b \\ b & c \end{pmatrix}$$

6.5.7 - Test to see if $R^T R$ is positive definite in each case:

$$R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and}$$

$$R = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

6.5.11 - Compute the three upper left determinants of A to establish the positive definiteness. Verify that their ratios give the second and third pivots.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$$

6.5.16 - A positive definite matrix cannot have a zero (or even worse, a negative number) on its diagonal. Show that this matrix fails to have $\mathbf{x}^T A\mathbf{x} > 0$:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ is not positive when } (x_1, x_2, x_3) =$$

6.6 - Similar Matrices

6.6.3 - Show that A and B are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

6.6.9 - By direct multiplication find A^2 and A^3 and A^5 when

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Guess the form of A^k. Set $k = 0$ to find A^0 and $k = -1$ to find A^{-1}.

6.6.10 - By direct multiplication, find J^2 and J^3 when

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Guess the form of J^k. Set $k = 0$ to find J^0. Set $k = -1$ to find J^{-1}.

6.6.12 - These Jordan matrices have eigenvalues $0, 0, 0, 0$. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For any matrix M, compare JM with MK. If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: J *is not similar* to K.

6.6.13 Based on Problem 12, what are the five Jordan forms when $\lambda = 0, 0, 0, 0$?

6.7 - Singular Value Decomposition (SVD)

6.7.1 - Find the eigenvalues and unit eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of $A^T A$. Then find $\mathbf{u}_1 = A\mathbf{v}_1/\sigma_1$:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \text{ and } A^T A = \begin{pmatrix} 10 & 20 \\ 20 & 40 \end{pmatrix} \text{ and } AA^T = \begin{pmatrix} 5 & 15 \\ 15 & 45 \end{pmatrix}.$$

Verify that $\mathbf{u}_1$ is a unit eigenvectors of AA^T. Complete the matrices U, Σ, V.

$$\text{SVD} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}^T.$$

6.7.4 - Find the eigenvalues and unit eigenvectors of A^TA and AA^T. Keep each $A\mathbf{v} = \sigma\mathbf{u}$:

Fibonacci Matrix $$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Construct the singular value decomposition and verify that A equals $U\Sigma V^T$.

6.7.6 - Compute $A^T A$ and AA^T and their eigenvalues and unit eigenvectors for this A:

Rectangular Matrix $$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Multiply the matrices $U\Sigma V^T$ to recover A. Σ has the same shape as A.

6.7.7 - What is the closest rank-one approximation to that 3 by 2 matrix?

6.7.9 - Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are orthonormal bases for $\mathbb{R}^n$. Construct the matrix A that transforms each $\mathbf{v}_j$ into $\mathbf{u}_j$ to give $A\mathbf{v}_1 = \mathbf{u}_1, \ldots, A\mathbf{v}_n = \mathbf{u}_n$.