Math 2270 - Assignment 11

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Section 6.1 - 2, 3, 5, 16, 17
Section 6.2 - 1, 2, 15, 16, 26
Section 6.4 - 1, 3, 5, 14, 23

6.1 - Introduction to Eigenvalues

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad A + I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

$A+I$ has the ___same___ eigenvectors as A. Its eigenvalues are ___increased___ by 1.

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = 3 - \lambda - 3\lambda + \lambda^2 - 8$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

Eigenvalues $\lambda = 5, -1$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ eigenvector}$$

So, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ has eigenvalues $\lambda = 5, -1$ w/ respective eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8 = 8 - 6\lambda + \lambda^2 - 8 = \lambda^2 - 6\lambda = \lambda(\lambda - 6)$$

Eigenvalues $\lambda = 6, 0$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ eigenvector}$$

So, $\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}$ has eigenvalues $\lambda = 6, 0$ w/ respective eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

6.1.3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace!

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

A^{-1} has the reciprocal eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues $1/\lambda_1,\ 1/\lambda_2$.

$$\begin{vmatrix} 0-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda)-2 = \lambda^2-\lambda-2 = (\lambda-2)(\lambda+1)$$

Eigenvalues $\lambda = 2, -1$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{vmatrix} -\frac{1}{2}-\lambda & 1 \\ \frac{1}{2} & -\lambda \end{vmatrix} = \left(-\tfrac{1}{2}-\lambda\right)(-\lambda)-\tfrac{1}{2} = \lambda^2+\tfrac{1}{2}\lambda-\tfrac{1}{2} = (\lambda+1)\left(\lambda-\tfrac{1}{2}\right)$$

Eigenvalues $\lambda = \frac{1}{2}, -1$

$$\begin{pmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ -1 \end{pmatrix} \text{ eigen vector}$$

So $\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 2, -1$, $\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$ has eigenvalues $\frac{1}{2}, -1$, and both have eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

6.1.5 Find the eigenvalues of A and B (easy for triangular matrices) and $A+B$:

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and}$$
$$A + B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Eigenvalues of $A+B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

A) $\begin{vmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) \Rightarrow$ eigenvalues $\lambda = 3, 1$

B) $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) \Rightarrow$ eigenvalues $\lambda = 3, 1$

A+B) $\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15$

$= (\lambda - 5)(\lambda - 3)$ eigenvalues $\lambda = 5, 3$

6.1.16 The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$. Start with the polynomial $det(A-\lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$
$$\text{so } det(A) = \underline{\lambda_1 \lambda_2 \cdots \lambda_n}.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

$$\begin{vmatrix} .8 & .3 \\ .2 & .7 \end{vmatrix} = .56 - .06 = .5 = \frac{1}{2} = (1)\left(\frac{1}{2}\right).$$

So, it checks out.

6.1.17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{has} \quad det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{})/2$ and $\lambda =$ __________. Their sum is $a+d$.
If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $det(A - \lambda I) =$ $(\lambda-3)(\lambda-4)$.

Eigenvalues $\lambda = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2a}$

$$= \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Sum is $\dfrac{a+d + \sqrt{(a-d)^2+4bc}}{2} + \dfrac{a+d - \sqrt{(a-d)^2+4bc}}{2} = a+d$

6.2 - Diagonalizing a Matrix

6.2.1 **(a)** Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

(b) If $A = S\Lambda S^{-1}$ then $A^3 = ()()()$ and $A^{-1} = ()()()$.

a) $\begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)$ Eigenvalues $\lambda = 1, 3$

$\lambda = 1$ $\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvector

$\lambda = 3$ $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector

$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$\begin{vmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$ $\lambda = 0, 4$

$\lambda = 0$ $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvector

$\lambda = 4$ $\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ eigenvector

$S = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$, $S^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

b)

$A^3 = S\Lambda^3 S^{-1}$

$A^{-1} = S\Lambda^{-1}S^{-1}$

6.2.2 If A has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 5$ with $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\mathbf{x}$'s.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad S^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 5 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$$

6.2.15 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than $\underline{\quad 1 \quad}$. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{pmatrix} .6 & .9 \\ .4 & .1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} .6 & .9 \\ .1 & .6 \end{pmatrix}.$$

(A_1) $\begin{vmatrix} -\lambda + .6 & .9 \\ .4 & -\lambda + .1 \end{vmatrix} = (.6-\lambda)(.1-\lambda) - 0.36$

$= \lambda^2 - .7\lambda - .30 = (\lambda - 1)(\lambda + .3)$

Eigen values $1, -.3$. So, $A_1^k \not\to 0$

Does not approach 0.

(A_2) $\begin{vmatrix} .6-\lambda & .9 \\ .1 & .6-\lambda \end{vmatrix} = (.6-\lambda)^2 - .09$

$= \lambda^2 - 1.2\lambda + .36 - .09 = \lambda^2 - 1.2\lambda + .27$

$= (\lambda - .9)(\lambda - .3)$

Eigenvalues $.9, .3$. So, $A_2^k \to 0$

Does approach 0.

6.2.16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the eigenvector of $\lambda = 1$.

$$\boxed{\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -.3 \end{pmatrix}} \qquad \begin{pmatrix} -.4 & .9 \\ .4 & -.9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} .9 \\ .4 \end{pmatrix}$$

$$\begin{pmatrix} .9 & .9 \\ .4 & .4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$S = \begin{pmatrix} .9 & 1 \\ .4 & -1 \end{pmatrix} \qquad S^{-1} = -\frac{1}{1.3}\begin{pmatrix} -1 & -1 \\ -.4 & .9 \end{pmatrix}$$

$$\lim_{k\to\infty} \begin{pmatrix} 1 & 0 \\ 0 & -.3 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \lim_{k\to\infty} S\Lambda^k S^{-1} = -\frac{1}{1.3}\begin{pmatrix} .9 & 1 \\ .4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -.4 & .9 \end{pmatrix}$$

$$= -\frac{1}{1.3}\begin{pmatrix} .9 & 1 \\ .4 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} = -\frac{1}{1.3}\begin{pmatrix} -.9 & -.9 \\ -.4 & -.4 \end{pmatrix}$$

$$= \frac{1}{1.3}\begin{pmatrix} .9 & .9 \\ .4 & .4 \end{pmatrix}$$

6.2.26 (Recommended) Suppose $A\mathbf{x} = \lambda\mathbf{x}$. If $\lambda = 0$ then $\mathbf{x}$ is in the nullspace. If $\lambda \neq 0$ then $\mathbf{x}$ is in the column space. Those spaces have dimensions $(n-r)+r=n$. So why doesn't every square matrix have n linearly independent eigenvectors?

The column space and nullspace are <u>not</u> orthogonal. The column space is orthogonal to the <u>left</u> nullspace, and the nullspace is orthogonal to the rowspace.

For example:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{has}$$

$$\vec{N}(A) = \vec{C}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

6.4 - Symmetric Matrices

6.4.1 Write A as $M+N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{pmatrix} = M + N \qquad (M^T = M, N^T = -N).$$

For any square matrix, $M = \frac{A+A^T}{2}$ and $N = \underline{\frac{A - A^T}{2}}$ add up to A.

$$A^T = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{pmatrix}$$

$$M = \frac{1}{2}\left(\begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{pmatrix}\right) = \begin{pmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{pmatrix}$$

$$N = \frac{1}{2}\left(\begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$$

6.4.3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda)(-\lambda) + 4\lambda + 4\lambda$$

$$= -\lambda^3 + 2\lambda^2 + 8\lambda = -\lambda(\lambda^2 - 2\lambda - 8)$$

$$= -\lambda(\lambda-4)(\lambda+2)$$

Eigenvalues $\lambda = 0, 4, -2$

$\lambda = 0$ $\quad \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{X}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \hat{X}_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\lambda = 4$ $\quad \begin{pmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{X}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \hat{X}_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$

$\lambda = -2$ $\quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{X}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \hat{X}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

6.4.5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda(\lambda+1)-4)+2(\lambda+1)2$$

$$= (1-\lambda)(\lambda^2+\lambda-4)+4(\lambda+1)$$

$$= \lambda^2+\lambda-4-\lambda^3-\lambda^2+4\lambda+4\lambda+4$$

$$= -\lambda^3+9\lambda = -\lambda(\lambda+3)(\lambda-3)$$

Eigenvalues $\lambda = 0, 3, -3$

$\lambda = 0$ $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \quad \hat{\vec{x}}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$\lambda = 3$ $$\begin{pmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad \hat{\vec{x}}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$\lambda = -3$ $$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \quad \hat{\vec{x}}_3 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{pmatrix}$$

6.4.14 (Recommended) This matrix M is skew-symmetric and also orthogonal. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($||M\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x}$ so $||\lambda\mathbf{x}|| = ||\mathbf{x}||$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \text{ can only have eigenvalues } i \text{ or } -i.$$

There must be repeated eigenvalues $i, i, -i, -i$ as

$0+0+0+0 = i+i-i-i = 0.$

6.4.23 (Recommended) To which of these classes do the matrices A and B belong: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Which of these factorizations are possible for A and B: $LU, QR, S\Lambda S^{-1}, Q\Lambda Q^T$?

A) Invertible, orthogonal, permutation, diagonalizable, Markov. (Not projection as $P^2 \neq P$).

$\lambda^2(1-\lambda)-1$

$\lambda^2-\lambda^3-1$

Factorizations: QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$

B) Diagonalizable, Projection, Markov.

Factorizations: LU, $S\Lambda S^{-1}$, $Q\Lambda Q^T$.