Math 2270 - Assignment 11

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Fall 2012

Section 6.1 - 2, 3, 5, 16, 17
Section 6.2 - 1, 2, 15, 16, 26
Section 6.4 - 1, 3, 5, 14, 23

6.1 - Introduction to Eigenvalues

6.1.2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad A + I = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}.$$

$A + I$ has the ________________________ eigenvectors as A. Its eigenvalues are ________________________ by 1.

6.1.3 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace!

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

A^{-1} has the ________________________ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ________________________.

6.1.5 Find the eigenvalues of A and B (easy for triangular matrices) and $A+B$:

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and}$$
$$A+B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Eigenvalues of $A+B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

6.1.16 The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$. Start with the polynomial $det(A-\lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$
$$\text{so } det(A) = \underline{\hspace{4cm}}.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

6.1.17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{has} \quad det(A - \lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{\ })/2$ and $\lambda =$ ________________________. Their sum is ________________________.
If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $det(A-\lambda I) =$ ________________________.

6.2 - Diagonalizing a Matrix

6.2.1 **(a)** Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

(b) If $A = S\Lambda S^{-1}$ then $A^3 = ()()()$ and $A^{-1} = ()()()$.

6.2.2 If A has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda_2 = 5$ with $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, use $S\Lambda S^{-1}$ to find A. No other matrix has the same λ's and $\mathbf{x}$'s.

6.2.15 $A^k = S\Lambda^k S^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than ________________________. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{pmatrix} .6 & .9 \\ .4 & .1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} .6 & .9 \\ .1 & .6 \end{pmatrix}.$$

6.2.16 (Recommended) Find Λ and S to diagonalize A_1 in Problem 15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $S\Lambda^k S^{-1}$? In the columns of this limiting matrix you see the ________________________.

6.2.26 (Recommended) Suppose $A\mathbf{x} = \lambda\mathbf{x}$. If $\lambda = 0$ then **x** is in the nullspace. If $\lambda \neq 0$ then **x** is in the column space. Those spaces have dimensions $(n - r) + r = n$. So why doesn't every square matrix have n linearly independent eigenvectors?

6.4 - Symmetric Matrices

6.4.1 Write A as $M + N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{pmatrix} = M + N \qquad\qquad (M^T = M, N^T = -N).$$

For any square matrix, $M = \frac{A+A^T}{2}$ and $N =$ ____________________ add up to A.

6.4.3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

6.4.5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}.$$

6.4.14 (Recommended) This matrix M is skew-symmetric and also ________________________.
Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($||M\mathbf{x}|| = ||\mathbf{x}||$ for every $\mathbf{x}$ so $||\lambda\mathbf{x}|| = ||\mathbf{x}||$ for eigenvectors.) Find all four eigenvalues from the trace of M:

$$M = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix} \text{ can only have eigenvalues } i \text{ or } -i.$$

6.4.23 (Recommended) To which of these classes do the matrices A and B belong: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Which of these factorizations are possible for A and B: $LU, QR, S\Lambda S^{-1}, Q\Lambda Q^T$?