Math 2270 - Assignment 10

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Section 5.2 - 1, 3, 11, 15, 16
Section 5.3 - 1, 6, 7, 8, 16

5.2 - Permutations and Cofactors

5.2.1 Compute the determinants of A, B, C from six terms.[1] Are their rows independent?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$|A| = 1 + 12 + 18 - 4 - 6 - 9 = 12$$

$$|B| = 28 + 40 + 72 - 24 - 56 - 60 = 0$$

$$|C| = 0 + 0 + 0 - 0 - 0 - 1 = -1$$

[1] Using the big formula.

5.2.3 Show that $det(A) = 0$ regardless of the five nonzeros marked by x's:

$$A = \begin{pmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{pmatrix}$$

What are the cofactors of row 1?

What is the rank of A?

What are the 6 terms in the big formula for $det(A)$?

$$C_{11} = \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0 \qquad C_{12} = -\begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0$$

$$C_{13} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

So, $det(A) = x\,0 + x\,0 + x\,0 = 0$.

$rank(A) = 2$ The first two columns are equal.

$$det(A) = 0 + 0 + 0 - 0 - 0 - 0 = 0$$

5.2.11 Find all the cofactors and put them into cofactor matrices C, D. Find AC and $det(B)$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{pmatrix}.$$

$$C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{pmatrix}$$

$$AC = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} ad-b^2 & ab-ac \\ cd-bd & ad-c^2 \end{pmatrix}$$

$$\det(B) = 7\,C_{31} = 7(-3) = \boxed{-21}$$

5.2.15 The tridiagonal $1, 1, 1$ matrix of order n has determinant E_n:

$$E_1 = |1| \qquad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \qquad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

(a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.

(b) Starting from $E_1 = 1$ and $E_2 = 0$ find $E_3, E_4, \ldots, E_8$.

(c) By noticing how these numbers eventually repeat, find E_{100}.

a) If we do a cofactor expansion on row 1 we get

$$E_n = 1 \cdot E_{n-1} - \begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & & & 1 & 1 \end{vmatrix}$$

For the second determinant we do a cofactor expansion down column 1 to get:

$$E_n = E_{n-1} - E_{n-2}$$

b) $E_3 = -1$, $E_4 = -1$, $E_5 = 0$, $E_6 = 1$, $E_7 = 1$, $E_8 = 0$

c) $E_{100} = E_{16(6)+4} = E_4 = \boxed{-1}$

5.2.16 F_n is the determinant of the $1, 1, -1$ tridiagonal matrix of order n:

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3$$

$$F_4 = \begin{vmatrix} 1 & -1 & & \\ 1 & 1 & -1 & \\ & 1 & 1 & -1 \\ & & 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* $1, 2, 3, 5, 8, 13, \ldots$. The sequence ususally starts $1, 1, 2, 3$ (with two 1's) so our F_n is the usual F_{n+1}.

$$F_n = \begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -1 & \cdots & 0 \\ 0 & 1 & 1 & -1 \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & 1 & 1 \end{vmatrix}$$

Cofactor expand along row 1 to get

$$= F_{n-1} + \begin{vmatrix} 1 & -1 & \cdots & & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ & 1 & 1 & -1 \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & 1 & 1 \end{vmatrix} = \cancel{F_{n}}$$

$F_{n-1} + F_{n-2}$

Cofactor expand on column 1

5.3 - Cramer's Rule, Inverses, and Volumes

5.3.1 Solve these linear equations by Cramer's rule $x_j = \frac{det(B_j)}{det(A)}$:

(a) $$\begin{aligned} 2x_1 &+ 5x_2 &= 1 \\ x_1 &+ 4x_2 &= 2 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}.$$

a) $A = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$ $\quad |A| = 8-5=3$

$B_1 = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$ $\quad |B_1| = 4-10=-6$

$B_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad |B_2| = 4-1=3$

$x_1 = -\frac{6}{3} = -2 \quad x_2 = \frac{3}{3} = 1$

b) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\quad |A| = 8-2-2=4$

$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\quad |B_1| = 3$ $\quad x_1 = \frac{3}{4}$

$x_2 = -\frac{1}{2}$

$B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad |B_2| = -2$

$x_3 = \frac{1}{4}$

$B_3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad |B_3| = 1$

5.3.6 Find A^{-1} from the cofactor formula $C^T/det(A)$. Use symmetry in part (b).

(a) $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 7 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

a) $C_{11} = 3 \quad C_{12} = 0 \quad C_{13} = 0$

$C_{21} = -2 \quad C_{22} = 1 \quad C_{23} = -7$

$C_{31} = 0 \quad C_{32} = 0 \quad C_{33} = 3$

$C = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{pmatrix} \quad C^T = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{pmatrix} \quad det(A) = 3$

$$\boxed{A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{pmatrix}}$$

b) $C_{11} = 3 \quad C_{12} = 2 \quad C_{13} = 1$

$C_{21} = 2 \quad C_{22} = 4 \quad C_{23} = 2$

$C_{31} = 1 \quad C_{32} = 2 \quad C_{33} = 3$

$C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$C^T = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$

$det(A) = 8 - 2 - 2 = 4$

$$\boxed{A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}}$$

5.3.7 If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?

If all cofactors are 0, then $C^T = 0$, where C^T is the zero matrix. If A had an inverse it would be $A^{-1} = \frac{C^T}{\det(A)} = 0$.

But, anything multiplied by the 0 matrix is 0, not I.

If none of the cofactors are 0, A could still be singular.

Take

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$C_{11} = 1$, $C_{12} = -2$

$C_{21} = -2$, $C_{22} = 4$

All non-zero, but A is not invertible.

5.3.8 Find the cofactors of A and multiply AC^T to find $det(A)$:

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

and $AC^T =$

If you change that 4 to 100, why is $det(A)$ unchanged?

$C_{21} = -(5-8) = +3 \quad C_{22} = 5-4 = 1 \quad C_{23} = -(2-1) = -1$

$C_{31} = 2-8 = -6 \quad C_{32} = -(2-4) = +2 \quad C_{33} = 2-1 = 1$

$$C = \begin{pmatrix} 6 & -3 & 0 \\ +3 & 1 & -1 \\ -6 & +2 & 1 \end{pmatrix} \qquad C^T = \begin{pmatrix} 6 & +3 & -6 \\ -3 & 1 & +2 \\ 0 & -1 & 1 \end{pmatrix}$$

$$AC^T = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 6 & +3 & -6 \\ -3 & 1 & +2 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

So, $det(A) = 3$

Use a cofactor expansion along row 1:

$1(6) + 1(-3) + 100(0) = 3$

This can be any number, still multiplying by 0.

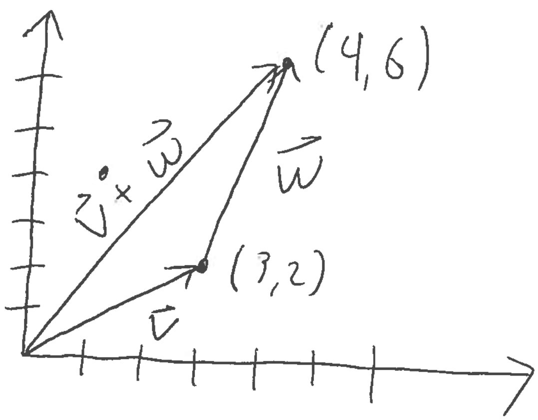
5.3.16 (a) Find the area of the parallelogram with edges $\mathbf{v} = (3, 2)$ and $\mathbf{w} = (1, 4)$.

(b) Find the area of the triangle with sides $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$. Draw it.

(c) Find the area of the triangle with sides $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{w} - \mathbf{v}$. Draw it.

a) $$\text{Area} = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12 - 2 = \boxed{10}$$

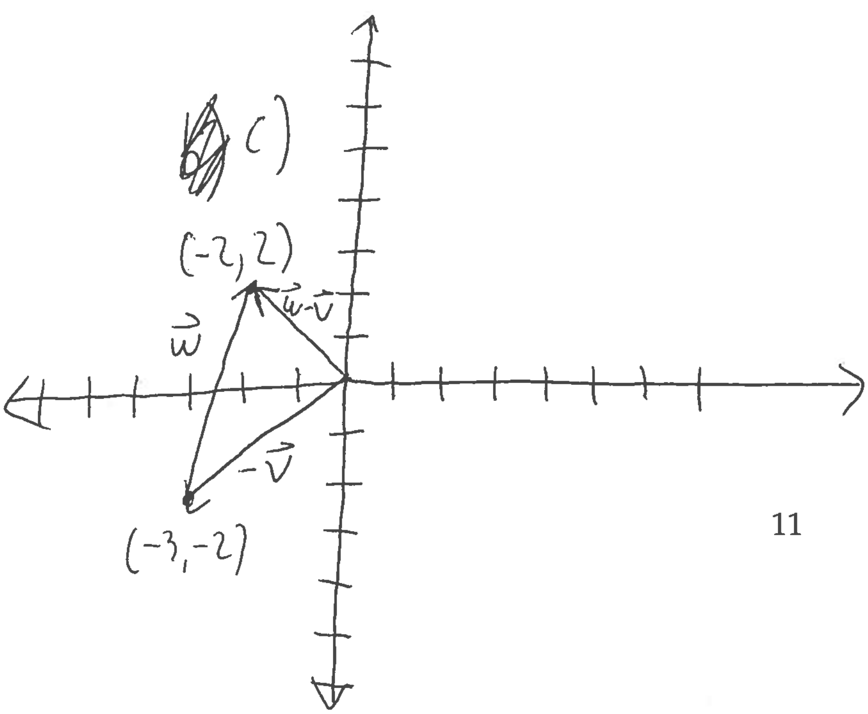
b)



$$\text{Area} = \frac{1}{2} \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix}$$

$$= \frac{1}{2}(10) = \boxed{5}$$

~~b~~ c)



$$\text{Area} = \frac{1}{2} \begin{vmatrix} -2 & 2 \\ -3 & -2 \end{vmatrix}$$

$$= \frac{1}{2}(4 - (-6))$$

$$= \frac{10}{2} = \boxed{5}$$