Math 1010 - Lecture 9 Notes

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Fall 2009

In today's lecture we'll extend our notions of solving algebraic equalities and inequalities to equations involving absolute values.

1 Absolute Value Equations

Suppose we're given the absolute value equation:

|x| = 12.

What this equation is asking, in words, is "for what numbers x is the absolute value of the number equal to 12?" In this case there will be two solutions, namely -12 and 12, both of which have absolute value 12.

This was pretty easy, but it illustrates this concept that, when dealing with an absolute value equation, we could possibly (and in fact frequently) have more than one solution.

Let's work through a slightly more difficult question. Suppose we're given the equation:

$$|x+3|+2=9$$

and we want to find the values of x for which this equation is true. In other words, the values of x that satisfy this equation. Well, to figure this out we'd first subtract 2 from both sides to get:

$$|x+3| = 7.$$

Now, this will be true if the absolute value of x + 3 is equal to 7. In other words, this will be true if x + 3 is equal to either 7 or -7. So, to find our solutions we'd set up:

$$|x+3| = 7$$

which means either
 $x+3 = 7$ or $x+3 = -7$.

So, to find our possible solutions, we'd solve each of these equations individually, to get x = 4 and x = -10, respectively. These *x*-values, 4 and -10, will be the two solutions to our equation.

It's frequently, but not always, the case that we get two solutions. We can have situations with more than two solutions¹, only one solution, or even zero solutions. For example, the equation:

$$|x| = 0$$

has only one solution, namely x = 0.

On the other hand, the equation:

$$|x| = -2$$

has **no** solution, as the absolute value of a number cannot be negative.

¹We won't see any of these in this section, as they'd involve equations that are more complicated than the ones we'll be examining.

Examples

Solve the following absolute value equations:

1. |5x - 3| + 8 = 22.

Solution

|5x - 3| + 8 = 22;

subtract 8 from both sides,

$$|5x-3| = 14;$$

So, either 5x - 3 = 14 or 5x - 3 = -14. We solve each of these individually.

5x - 3 = 14;

add 3 to both sides,

5x = 17;

divide both sides by 5,

$$x = \frac{17}{5}.$$

The other possibility is:

5x - 3 = -14;

add 3 to both sides,

5x = -11;

divide both sides by 5,

$$x = -\frac{11}{5}.$$

So, the two possible solutions are
$$x = \frac{17}{5}$$
 or $x = -\frac{11}{5}$.

2. |4 - 3x| = 0.

Solution

Here there's only one possible solution, namely 4 - 3x = 0, because 0 is its own opposite. Solving this equation we get:

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4 - 3x = 0;
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add 3x to both sides,

4 = 3x;

divide both sides by 3,

$$\frac{4}{3} = x.$$

So, the only solution is $x = \frac{4}{3}$.

3. |5 - 2x| + 10 = 6.

Solution

If we subtract 10 from both sides of this equation we get:

$$|5-2x| = -4.$$

This has no solutions, as an absolute value cannot be negative.

2 Equations with Absolute Values on Both Sides

So far we've just dealt with equations with one absolute value. We can also deal with equations with more than one absolute value. For example, suppose we're given the equation:

$$|10 - 3x| = |x + 7|.$$

This will be true if either 10 - 3x = x + 7 or 10 - 3x = -(x + 7). So, the two equations could be equal, or they could be *opposites*. In either case, they would have the same absolute value.

So, to solve this we'd need to break it up into its two cases.

Case I - Both sides are equal

10 - 3x = x + 7;

add 3x to both sides,

$$10 = 4x + 7$$

subtract 7 from both sides,

$$3 = 4x;$$

divide both sides by 4,

$$\frac{3}{4} = x.$$

Case II - The sides are opposite

$$10 - 3x = -(x + 7)$$

 $\rightarrow 10 - 3x = -x - 7;$

add 3x to both sides,

$$10 = 2x - 7;$$

add 7 to both sides,

$$17 = 2x;$$

divide both sides by 2,

$$\frac{17}{2} = x.$$

So, our two solutions (the two values of *x* that satisfy the equation) are x = 3/4 and x = 17/2.

Examples

Solve for *x* in the following equations.

1. 3|2x-5|+4=7.

Solution

First, we subtract 4 from both sides to get,

3|2x-5|=3;

then if we divide both sides by 3 we get,

$$|2x-5| = 1.$$

There are two possible solutions here, corresponding to 2x - 5 = 1 and 2x - 5 = -1. The solutions in these cases (using the same algebraic techniques as always) are x = 3 and x = 2.

2. |2x+7| = |2x+9|.

Solution

This equality will be satisfied if either 2x + 7 = 2x + 9 or 2x + 7 = -(2x + 9). In the first case we have:

$$2x + 7 = 2x + 9;$$

subtracting 2x from both sides we get,

7 = 9.

As this is never true, there are no solutions to this case.

For the other case we have:

2x + 7 = -2x - 9;

add 2x to both sides we get,

4x + 7 = -9;

subtracting 7 from both sides we get,

4x = -16;

dividing both sides by 4 we get,

x = -4.

So, there's only one solution to this equations, and it's x = -4.

3 Absolute Value Inequalities

Thus far we've only dealt with equalities involving absolute values. We can also have inequalities with absolute values.

For example, the possible values of x that satisfy the inequality below are plotted on the number line:



In general, the solutions of |x| < a are all values of x that lie between -a and a. In other words, the values of x such that -a < x < a.

On the other hand, the solutions of |x| > b are all values of x that are greater than b or less than -b. In other words, the values of x such that either x > b or x < -b.

We note that the above relations hold if we replace < with \leq , and/or > with \geq .

Examples

Solve the following inequalities, and graph the possible solutions on the number line.

1. $|7 - 2h| \ge 9$.

Solution

This will be true if $7 - 2h \ge 9$ or $7 - 2h \le -9$. Solving for *h* in these two cases we get $-1 \ge h$ or $8 \le h$. Graphing these possibilities on the number line we have:



2. |3x + 10| < -1.

Solution

This is never satisfied, as any number less than -1 must be negative, and the absolute value of a number cannot be negative. So, there are no solutions.

3.
$$\left|\frac{2x-4}{5}\right| - 9 \le 3.$$

Solution

If we add 9 to both sides of this equation we get:

$$\left|\frac{2x-4}{5}\right| \le 12.$$

This will be satisfied if $-12 \le \frac{2x-4}{5} \le 12$. Multiplying all terms here by 5 we get:

 $-60 \le 2x - 4 \le 60;$

adding 4 to all terms we get,

$$-56 \le 2x \le 64;$$

dividing all terms by 2 we get,

$$-28 \le x \le 32.$$

So, our inequality is satisfies for $-28 \le x \le 32$.

