Math 1010 - Lecture 31 Notes

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In this, our final lecture, we'll go over some more properties of logarithms, and then we'll work through some problems involving logarithms.

1 Properties of Logarithms

We saw in our last lecture that the properties of exponentials:

(a) $a^0 = 1$

(b) $a^1 = a$

(c) $a^x = a^x$

corresponded to the properties of logarithms:

(a) $\log_a(1) = 0$

(b) $\log_a(a) = 1$

(c) $\log_a(a^x) = x$.

Well, similarly, the properties of exponentials:

(d) $a^m a^n = a^{m+n}$

(e) $\dfrac{a^m}{a^n} = a^{m-n}$

(f) $(a^m)^n = a^{mn}$

correspond to the properties of logarithms:

(d) $\log_a(uv) = \log_a(u) + \log_a(v)$

(e) $\log_a\left(\dfrac{u}{v}\right) = \log_a(u) - \log_a(v)$

(f) $\log_a(u^n) = n\log_a(u)$.

Therefore, the six big properties of logarithms are:

(a) $\log_a(1) = 0$

(b) $\log_a(a) = 1$

(c) $\log_a(a^x) = x$

(d) $\log_a(uv) = \log_a(u) + \log_a(v)$

(e) $\log_a\left(\dfrac{u}{v}\right) = \log_a(u) - \log_a(v)$

(f) $\log_a(u^n) = n\log_a(u)$.

Note that the first three can be derived from the last three, but they're so common and important that we single them out for notice.

Note that it is *not* in general possible to simplify $\log_a(u+v)$ and it is certainly *not* the case that this is equal to $\log_a(u) + \log_a(v)$. So, don't make this mistake!

2 Using the Properties

Examples

Use the properties of logarithms to evaluate the following.

1. $\log_5\left(\sqrt[3]{5}\right)$

$$= \frac{1}{3}\log_5(5) = \boxed{\frac{1}{3}}$$

2. $\log_3(54) - \log_3(2)$

$$= \log_3(2(27)) - \log_3(2)$$
$$= \log_3(27) + \log_3(2) - \log_3(2) = \log_3(3^3) = 3\log_3(3)$$
$$= \boxed{3}$$

3. $\log_4\left(\frac{3}{16}\right) + \log_4\left(\frac{1}{3}\right)$

$$= \log_4(3) - \log_4(16) + \log_4(1) - \log_4(3)$$
$$= -\log_4(16) = \boxed{-2}$$

4. $\ln\left(\frac{e^3}{e^2}\right)$

$$= \ln(e^3) - \ln(e^2) = 3 - 2 = \boxed{1}$$

Use the properties of logarithms to *expand* the expressions.

1. $\log_3(11x)$

$$= \log_3(11) + \log_3(x)$$

2. $\log_4[x^6(x+y)^2]$

$$= 6\log_4(x) + 2\log_4(x+y)$$

3. $\ln(\sqrt{x(x+2)})$

$$= \frac{1}{2}\left[\ln(x) + \ln(x+2)\right]$$

Use the properties of logarithms to *contract* the expressions.

1. $\log_3(2) + \frac{1}{2}\log_3(y)$

$$= \log_3(2\sqrt{y})$$

2. $2[\ln(x) - \ln(x+1)]$

$$= 2\ln\left(\frac{x}{x+1}\right) = \boxed{\ln\left(\left(\frac{x}{x+1}\right)^2\right)}$$

3. $\frac{1}{4}\log_6(x+1) - 5\log_6(x-4)$

$= \log_6\left(\sqrt[4]{x+1}\right) - \log_6\left((x-4)^5\right)$

$= \boxed{\log_6\left(\frac{\sqrt[4]{x+1}}{(x-4)^5}\right)}$

3 Problems involving Logarithms

1. If you invest \$1000 in a savings account that generates 2% interest per year, compounded annually, how long will it takes for your investment to quintuple (increase to fives times its original amount)? Express your answer as a logarithm, you don't need to calculate it.

$\$5000 = \$1000\,(1.02)^t$

$\Rightarrow \quad 5 = (1.02)^t \Rightarrow \log_{10}(5) = t\log(1.02)$

$\Rightarrow \boxed{t = \frac{\log_{10}(5)}{\log_{10}(1.02)}} = \boxed{81.27 \text{ years}}$

2. Solve the exponential equation $5000 = 2500e^{0.09t}$ for t to determine the number of years for an investment of \$2500 to double in value when compounded continuously at the rate of 9%.

$2 = e^{0.09t}$

$\Rightarrow \ln 2 = 0.09t$

$\Rightarrow \boxed{\frac{\ln 2}{0.09} = t}$ Note: $\boxed{7.7 \text{ years}}$

3. Use Newton's Law of Cooling to solve the following forensics problem. Newton's law of cooling states that:

$$kt = \ln\left(\frac{T-S}{T_0-S}\right)$$

where T is the temperature of a body (in degrees Fahrenheit), t is the number of hours elapsed, S is the temperature of the environment, and T_0 is the initial temperature of the body.

A corpse was discovered in a motel room at 10:00 P.M., and its temperature was $85°F$. Three hours later, the temperature of the corpse was $78°F$. The temperature of the motel room is a constant $65°F$.

(a) What is the constant k?

$$Kt_o = \ln\left(\frac{85-65}{T_o-65}\right) \qquad K(t_o+3) = \ln\left(\frac{78-65}{T_o-65}\right)$$

$$\Rightarrow 3k = \ln(78-65) - \ln(85-65)$$

$$k = \frac{1}{3}\ln\left(\frac{13}{20}\right) = -0.1436$$

(b) Find the time of death using the fact that the temperature of the corpse at the time of death was $98.6°F$.

$$kt_o = \ln\left(\frac{98.6-65}{98.6-65}\right) = 0 \Rightarrow t_o = 0.$$

$$kt = \ln\left(\frac{85-65}{98.6-65}\right) = -0.51879$$

$$\Rightarrow t = 3.613 \text{ hours} \Rightarrow \quad 10{:}00PM - 3{:}37 \Rightarrow \boxed{6{:}23\ PM}$$

(c) What is the temperature of the corpse two hours after death?

$$2k = \ln\left(\frac{T-S}{T_o-S}\right)$$

$$\Rightarrow 2k + \ln(t_o - S) = \ln(T-S)$$

$$\Rightarrow e^{(2k+\ln(T_o-S))} = T-S \Rightarrow T = S + e^{(2k+\ln(T_o-S))}$$

$$\Rightarrow T = 65 + e^{(2(-0.1436)+\ln(98.6-65))} = \boxed{90.2°F}$$