

Math 1010 - Lecture 30 Notes

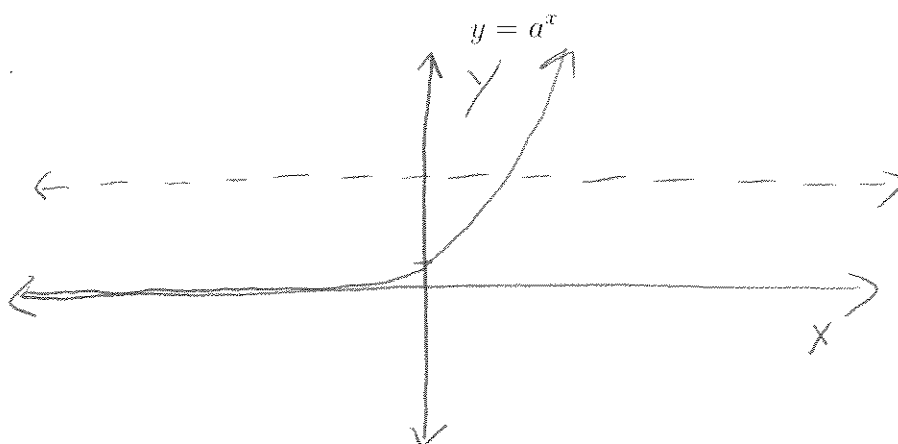
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Suppose you wanted to answer the question “for what number x is 2^x equal to 8?” Or, put more mathematically, you want to solve the equation:

$$8 = 2^x.$$

Now, in this case we might be able to eyeball it and see that the answer is 3, but we’d like a more precise way of figuring this out. So, to get this more precise way we first sketch a graph of the exponential function:



If we take a look at this graph we note that it passes the horizontal line test. In other words, any horizontal line that we draw on the graph only hits the graph in at most one point. So, we can invert the function.

If we try to invert the exponential using our standard method we run into a problem:

$$y = a^x,$$

switch the x and y terms,

$$x = a^y$$

and then what? How can we isolate y ? Well, we don't know how to do that yet because we've never dealt with this type of function before, but the answer to this is that we have to take a logarithm of both sides. In particular, a logarithm with a base of a :

$$\log_a(x) = y.$$

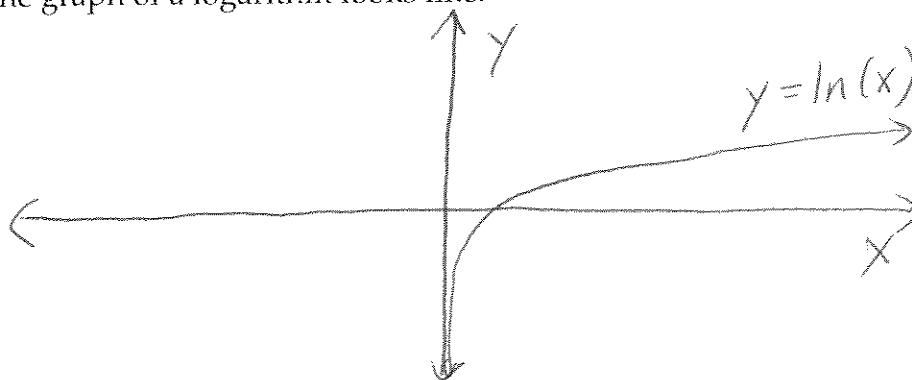
What is a logarithm? I'm glad you asked. A logarithm is the inverse of an exponential function. To be more explicit let a and x be positive real numbers such that $a \neq 1$. The *logarithm of x with base a* is denoted by $\log_a(x)$ and is defined as:

$$y = \log_a(x) \text{ if and only if } x = a^y.$$

The function $f(x) = \log_a(x)$ is the *logarithmic function with base a* .

So, $y = \log_a(x)$ is equivalent to $x = a^y$.

The graph of a logarithm looks like:



Note that the domain of a logarithm is all positive numbers, a.k.a. $x > 0$.

Let's do some examples that might make this more concrete.

Examples

1. $\log_2(16)$

$$16 = 2^4 \Rightarrow \log_2(16) = \boxed{4}$$

2. $\log_3(9)$

$$9 = 3^2 \Rightarrow \log_3(9) = \boxed{2}$$

3. $\log_{10}(1000)$

$$1000 = 10^3 \Rightarrow \log_{10}(1000) = \boxed{3}$$

4. $\log_2(1/8)$

$$\frac{1}{8} = \frac{1}{2^3} = 2^{-3} \Rightarrow \log_2(1/8) = \boxed{-3}$$

5. $\log_7(1)$

$$1 = 7^0 \Rightarrow \log_7(1) = \boxed{0}$$

This last example is one particular case of a general rule for logarithms. Namely, the first rule below.

Properties of Logarithms

- $\log_a(1) = 0$ because $a^0 = 1$.
- $\log_a(a) = 1$ because $a^1 = a$.
- $\log_a(a^x) = x$ because $a^x = a^x$.

Now, on most calculators there are two logarithm buttons.¹ One is *LOG* or *log*, which means log base 10, and the other is *LN* or *ln*, which is called the natural logarithm, and which means log base e .²

Now, with only these two buttons how would we calculate, say $\log_7(20)$? I'm glad you ask. The way we do this is with the change of base formula:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

where a and b are positive real numbers and $a, b \neq 1$. So, we can use just one logarithm button to calculate any logarithm we want. To calculate $\log_7(20)$ we could use the $\ln$ button and calculate this as:

$$\log_7(20) = \frac{\ln(20)}{\ln(7)} \approx 1.5395.$$

¹This is one place where it's very nice to have a calculator. Logarithms are, in general, hard to calculate precisely. For example, what is $\log_{3.297}(5.291)$? That's hard to figure out. So, before calculators, there were big tables of logarithms that people would have to turn to in order to look these values up. They'd usually take up about ten pages in the back of the algebra book, if not more. Also, the slide ruler, which was the mechanical precursor to the handheld electronic calculator, was based upon the use of logarithms to aid calculations. The first table of logarithms was put together in 1614 by John Napier. He had the table long before that, but delayed publishing it because he knew it could be used to make more precise cannons, and didn't want those improved cannons used by the Turks against Christians. He also calculated that, according to the Bible, the end of the world would happen in the year 1688. His table of logarithms was more accurate. One might even say that the cannons based upon his work were more accurate than the canons based upon his work! What, no laughs? Anybody? Anybody? Is this microphone on?

² e is the number we talked about in lecture 29. It's approximately 2.71828.

Example

Calculate $\log_9(27)$

$$\log_9(27) = \frac{\log_3(27)}{\log_3(9)}$$

$$27 = 3^3$$

$$9 = 3^2$$

$$\Rightarrow \frac{\log_3(27)}{\log_3(9)} = \boxed{\frac{3}{2}}$$