

Math 1010 - Lecture 29 Notes

Dylan Zwick

Fall 2009

A function $f(x)$ is a rule that assigns a given input number to a given output number. The set of possible inputs is the *domain* of the function, while the set of possible outputs is the *range*. For it to be a function, there must be only one output for a given input, although more than one input can have the same output.

Now, suppose we have two functions:

$$\begin{aligned}f(x) &= x + 1 \\g(x) &= x^2.\end{aligned}$$

We can combine these functions to form a *composite* function. The way we do this is we take the output from one, and make it the input to the other. We write this as $(g \circ f)(x)$ or as $g(f(x))$.

For example, if $x = 2$ then we have $f(x) = 2 + 1 = 3$. We take the output from $f(x)$, and plug it in as the input for $g(x)$. So, $g(f(2)) = g(3) = 3^2 = 9$. If we were to write $g(f(x))$ as a function in it's own right it would be:

$$g(f(x)) = (f(x))^2 = (x + 1)^2 = x^2 + 2x + 1.$$

Note that it is not the case in general that $g \circ f = f \circ g$. In our particular case $f \circ g$ would be:

$$f(g(x)) = f(x^2) = x^2 + 1.$$

Obviously $x^2 + 2x + 1$ and $x^2 + 1$ are two different functions.

Examples

For the functions:

$$f(x) = \frac{1}{1+x}$$

and

$$g(x) = x^2 + 2$$

determine:

1. $f(g(2))$

$$g(2) = 2^2 + 2 = 6$$

$$f(6) = \frac{1}{1+6} = \boxed{\frac{1}{7}}$$

2. $g(f(2))$

$$f(2) = \frac{1}{1+2} = \frac{1}{3}$$

$$g\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + 2 = \frac{1}{9} + 2 = \boxed{\frac{19}{9}}$$

3. $f(g(x))$

$$f(g(x)) = \frac{1}{1+g(x)} = \frac{1}{1+(x^2+2)} = \boxed{\frac{1}{3+x^2}}$$

4. $g(f(x))$

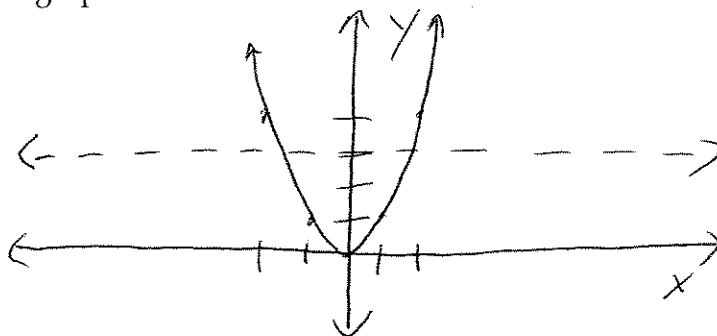
$$g(f(x)) = (f(x))^2 + 2$$

$$= \left(\frac{1}{1+x}\right)^2 + 2 = \frac{1}{(1+x)^2} + \frac{2(1+x)^2}{(1+x)^2} = \frac{2x^2 + 4x + 3}{x^2 + 2x + 1}$$

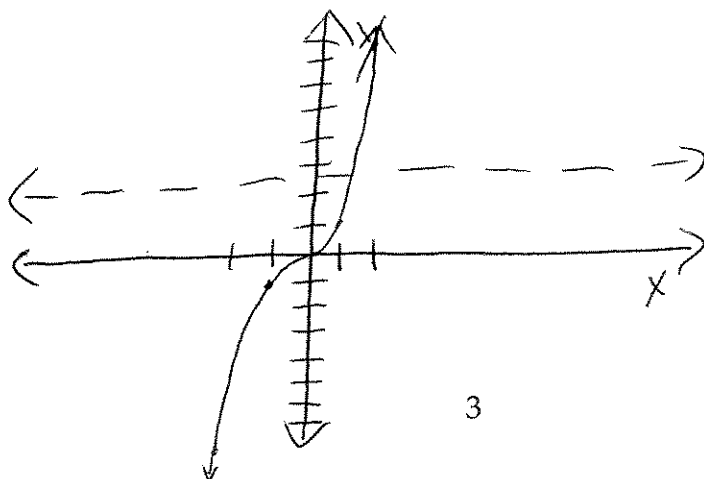
If a function is such that every output comes from a unique input, then it makes sense to ask the question "for a given output, what is the input that generated it." Functions for which this question has an answer are called invertible functions, and the function that answers this question is called the inverse function.

A quick, graphical way that we can test for whether a function is invertible is something called the horizontal line test. If you can draw a horizontal line that hits more than one point in the graph of your function, then your function is not invertible. If you can't, then it is. For example, if we examine the graphs of the functions:

$f(x) = x^2$



$g(x) = x^3$



we can see that $f(x)$ fails the horizontal line test, while $g(x)$ passes. This is because x^2 is not invertible,¹ while x^3 is.²

Formally, if f and g are two functions such that

$$f(g(x)) = x \quad \text{for every } x \text{ in the domain of } g,$$

and

$$g(f(x)) = x \quad \text{for every } x \text{ in the domain of } f,$$

then the function g is called the *inverse function* of the function f , and is denoted by f^{-1} . Note that the inverse of the function f^{-1} is just f .

If we're given a function and we want to find its inverse, we can find it by following these steps:

Step 1 In the equation for $f(x)$, replace $f(x)$ with y .

Step 2 Interchange x and y .

Step 3 If the new equation does not represent y as a function of x , the function f does not have an inverse function. If the new equation does represent y as a function of x , solve the new equation for y .³

Step 4 Replace y with $f^{-1}(x)$.

Step 5 Verify that f and f^{-1} are inverse functions of each other by showing that $f(f^{-1}(x)) = x = f^{-1}(f(x))$.⁴

¹For example, if you're told $x^2 = 4$ you don't know if $x = 2$ or if $x = -2$.

²If you're told that $x^3 = -8$ then the only real number that x could possibly be is -2 .

³If the function does not have an inverse, then it will fail the horizontal line test. So, if you've done the horizontal line test and verified that your function has an inverse, you can skip the verification and just solve for y .

⁴You don't technically have to do this. This is just a way of checking your answer.

Let's work some examples:

Examples - Find the inverses of the following functions.

1. $f(x) = 2x + 3$

$$y = 2x + 3$$

$$\Rightarrow x = 2y + 3$$

$$\Rightarrow \frac{x-3}{2} = y$$

$$f^{-1}(x) = \frac{x-3}{2}$$

2. $f(x) = x^3 + 3$

$$y = x^3 + 3$$

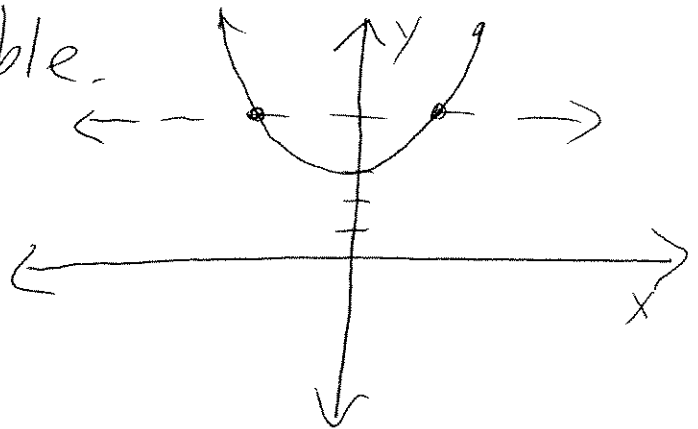
$$\Rightarrow x = y^3 + 3$$

$$\Rightarrow x - 3 = y^3$$

$$y = \sqrt[3]{x-3}$$

3. $f(x) = x^2 + 3$

Not invertible.



$$4. f(x) = \sqrt{x^2 - 4} \quad x \geq 2.$$

$$y = \sqrt{x^2 - 4}$$

$$\Rightarrow x = \sqrt{y^2 - 4}$$

$$x^2 = y^2 - 4$$

$$\Rightarrow x^2 + 4 = y^2$$

$$\Rightarrow y = \sqrt{x^2 + 4}$$

$$f^{-1}(x) = \sqrt{x^2 + 4}$$

$$x \geq 0.$$