

Math 1010 - Lecture 28 Notes

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In his will Benjamin Franklin left \$1000 to both the city of Philadelphia and the city of Boston, with the instructions that the money should not be spent immediately, but instead invested at an annual interest rate of 5%. In 100 years both cities would then be allowed to spend three-fourths of the money, but would be required to reinvest the remaining one-fourth for another 100 years. At the conclusion of 200 years, the money could then be used by the city in whatever way it saw fit.

Franklin figured that in 100 years there would be \$131,501 in the account, and in 200 years there would be \$4,323,104.

Now, 5% interest proved hard to come by every year for 200 years, but when the trust finally came due about 20 years ago Philadelphia had about \$2,000,000, while Boston had about \$5,000,000!¹ Philadelphia has used the money for scholarships for high school students, while Boston has used the money to found a trade school.

The reason this relatively small amount of seed money was able to grow so large is because of a phenomenon called *compound interest*, which Einstein called, jokingly, the most powerful force in the universe. Franklin summed up the idea pithily as “the money that money makes, makes money.” Mathematically, compound interest is an example of something called an exponential function. We’ll be studying these for our last two weeks of class. You’ll study them much more if you go on to math 1030.

¹So, invest in Boston!

1 Exponential Functions

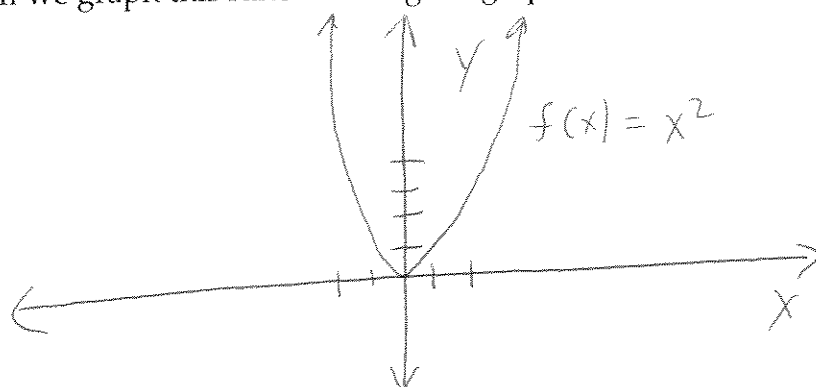
We've seen constant exponentials many times before in this class. For example, the function:

$$f(x) = x^2$$

has the exponent 2, and some sample values:

$$\begin{aligned} f(1) &= 1 & f(2) &= 4 & f(3) &= 9 \\ f(4) &= 16 & f(5) &= 25 & f(6) &= 36. \end{aligned}$$

If we graph this function we get a graph that looks like:



An exponential function is a function where the base of the exponential is constant, while the exponent itself is variable. For example, the function:

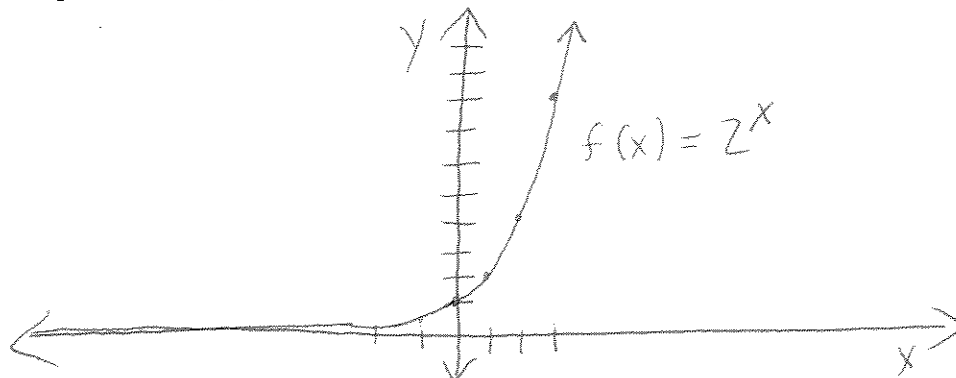
$$g(x) = 2^x.$$

Some sample values for this function are:

$$\begin{aligned} g(1) &= 2 & g(2) &= 4 & g(3) &= 8 \\ g(4) &= 16 & g(5) &= 32 & g(6) &= 64. \end{aligned}$$

Now, we note that our function g is getting larger than our function f . This is a trend that will accelerate. For example $f(10) = 100$, while $g(10) = 1024$. Going out a bit farther we'd get $f(20) = 400$, while $g(20)$ would be over one million. Exponential functions get large fast!

A part of the graph of our exponential function would look like this:



This shape is typical. In one direction the curve will get very close to the x -axis. This is an example of something called an asymptote. While in the other direction the graph will get very big, very fast.

Formally an exponential function is defined as:

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.¹ So, the domain of $f(x)$ here would be all real numbers.

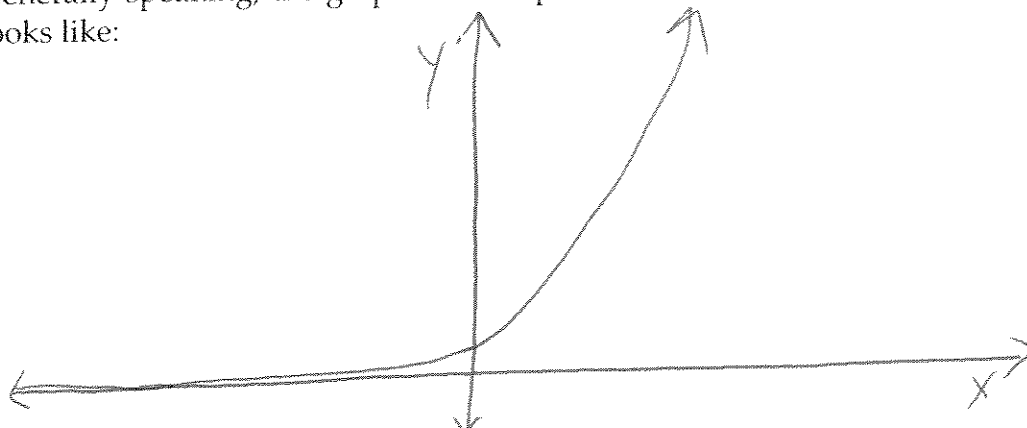
Now, exponential functions follow the same exponential rules we've all gotten to know and love:

- $a^x \cdot a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$

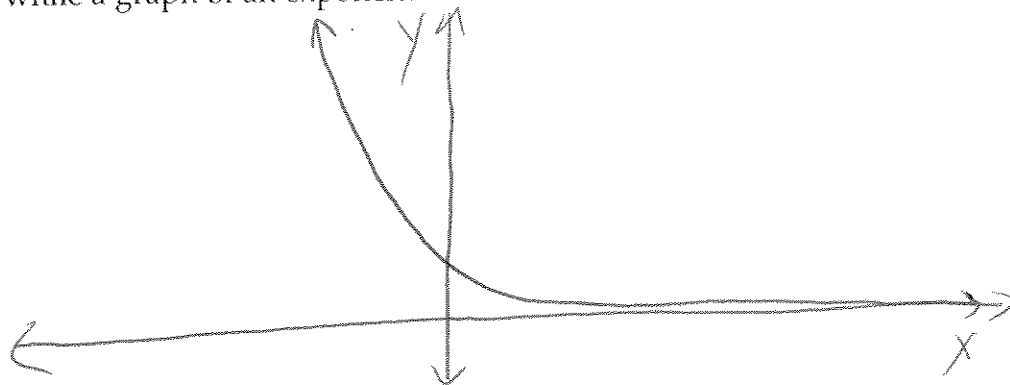
¹Note that if $a = 1$ this would just be the constant function 1.

2 Graphing Exponential Functions

Generally speaking, the graph of an exponential function of the form a^x looks like:



while a graph of an exponential function of the form a^{-x} looks like:



The graph of a^{x+c} is the graph of a^x shifted c units to the left, while the graph of $a^x + c$ is the graph of a^x shifted c units up.

Examples

Sketch the graph the following functions:

1. 3^x

2. 3^{-x}

3. 3^{x+1}

4. $3^x - 4$

Now, there's a very common base for an exponential. This number is so common that we've given it its own letter. We denote this number by e .

The number e is irrational³ and it's approximately:

$$e \approx 2.71828$$

The precise definition of e , and why it's important, is not a subject for this class.⁴ However, you should know it exists, and it will come up in relation to the $\ln$ button on your calculator.

3 Compound Interest

After t years, the balance A in an account with principal (starting amount) P and annual interest rate r (in decimal form) is given by one of the following formulas:

- For n compoundings per year: $A = P \left(1 + \frac{r}{n}\right)^{nt}$.
- For continuous compounding: $A = Pe^{rt}$.⁵

Here A is an exponential function of t .

³Transcendental, in fact.

⁴It's a subject for a calculus class. Just in case you're curious, there are many equivalent ways of precisely defining e . One way is as:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

As for why the number comes up so much, the fundamental reason is that it's very involved in the answer to the question "what function, if any, is equal to its derivative?" If you'd like this to make sense to you, I'd very much encourage you to take calculus.

⁵See, I told you it comes up. What continuous compounding means is that we make the number of times it compounds each year larger, and larger, and larger, and then take the limit as this number goes to infinity. If you want to understand that more precisely, take a calculus class.

Example

Calculate the amount of money you'd expect to receive on an investment of \$1,000 dollars at 5% annual interest after 200 years if the interest is compounded:

1. Annually.⁶

2. Monthly.

3. Continuously.

⁶This is how much Franklin would have calculated the trust to be worth in 200 years if three fourths were not taken out after 100 years.