

Math 1010 - Lecture 27 Notes

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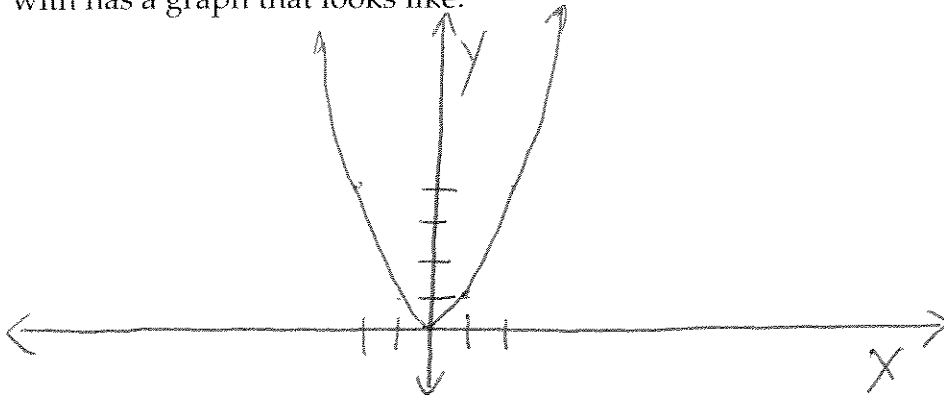
The graph of a quadratic equation is the graph of a function of the form:

$$y = ax^2 + bx + c \quad a \neq 0.$$

A simple, and perhaps the simplest, example of this is the equation:

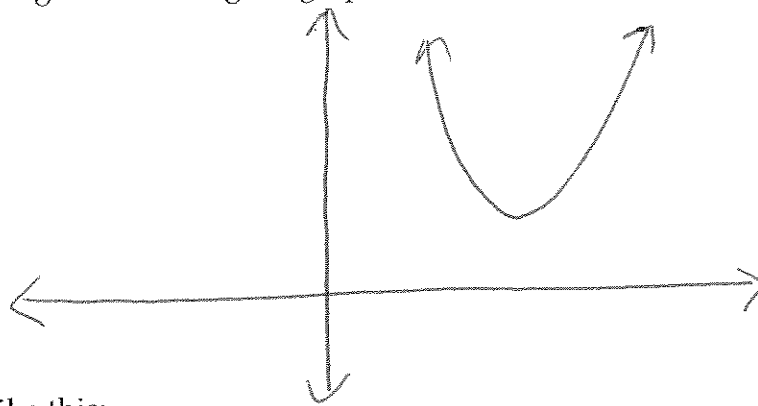
$$y = x^2;$$

with has a graph that looks like:

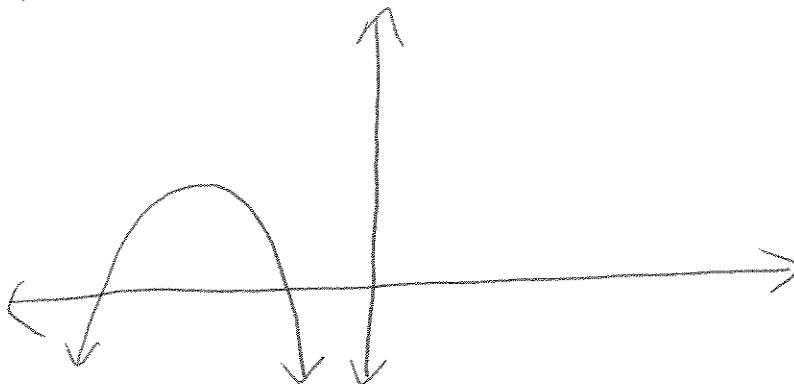


The shape of this graph is a parabola, and in fact all graphs of quadratic equations are parabolas like this one. The only difference is they might be shifted around, stretched or squeezed, or flipped.

So, in general we'll get a graph that looks like this:



or like this:



The *vertex* of the parabola is either the lowest point (in the first picture) or the highest point (in the second picture). The vertical line through the vertex is called the *axis* of the parabola, and the parabola is symmetric around its axis. Finally, note that the x -intercepts of the parabola are the roots of the corresponding quadratic equation.

1 Standard form, finding the vertex, and graphing a parabola based upon its equation.

We call the *standard form* of a quadratic function the function we get after we complete the square. So, it will be a function of the form:

$$f(x) = a(x - h)^2 + k.$$

If we have a quadratic equation in this form, then the vertex will be at the point (h, k) .

If we're just given a quadratic equation:

$$f(x) = ax^2 + bx + c$$

then we can find the vertex either by completing the square, or by using the formula that the x -value of the vertex is at the point $x = -b/(2a)$.¹

Finally, we can plot a parabola from its equation by noting a few basic facts:

- If $a > 0$ then the parabola opens up, if $a < 0$ then the parabola opens down.
- The vertex of the parabola, if our equation is just a quadratic, is at $x = -b/(2a)$. If it's in completed square form, a.k.a. standard form, then it's at the point (h, k) .
- The parabola is symmetric about its axis.

So, if we want to graph a parabola, we first figure out if it opens up or down, find the vertex, plot some points, and then play connect the dots keeping in mind that the parabola is symmetric about its axis.

¹Note that we could figure this out by following our general completing the square process we used to get the quadratic formula.

Examples

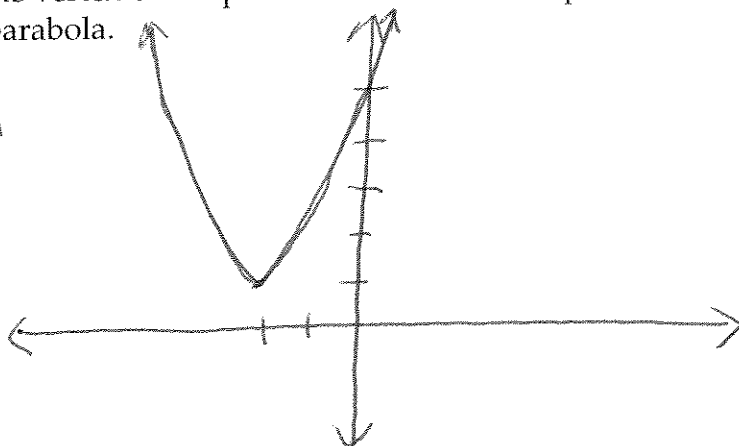
1. Find the standard form of the function:

$$f(x) = x^2 + 4x + 5,$$

and use it to calculate the vertex of the parabola the function represents. Then, graph the parabola.

$$f(x) = (x+2)^2 + 1$$

$$\text{Vertex: } (-2, 1)$$



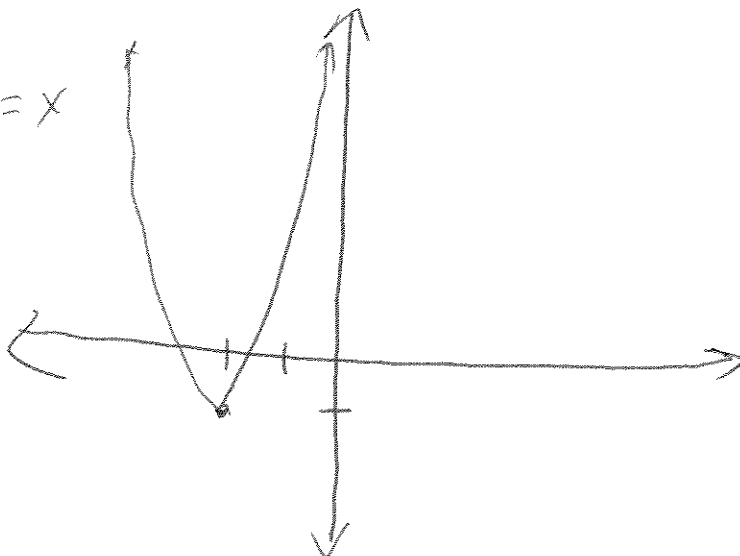
2. Find the vertex of the parabola given by the function:

$$f(x) = 2x^2 + 8x + 7$$

and use it to graph the parabola.

$$\text{Vertex: } -\frac{8}{2(2)} = -2 = x$$

$$\begin{aligned} f(-2) &= 2(-2)^2 + 8(-2) + 7 \\ &= 8 - 16 + 7 = -1 \end{aligned}$$



3. Find the equation of the parabola with vertex $(-2, 1)$ and y -intercept $(0, -3)$. Graph the parabola.

$$f(x) = a(x-h)^2 + k$$

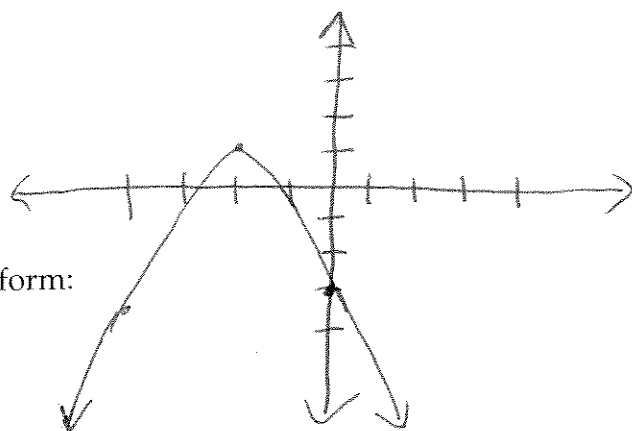
$$h = -2, k = 1$$

$$\begin{aligned} f(x) &= a(x+2)^2 + 1 \\ &= ax^2 + 4ax + 4a + 1 \end{aligned}$$

$$f(0) = 4a + 1 = -3$$

$$\Rightarrow 4a = -4 \Rightarrow a = -1$$

$$\text{So, } \boxed{f(x) = -(x+2)^2 + 1}$$



1.1 Polynomial Inequalities

A polynomial inequality is an inequality of the form:

$$f(x) \geq k$$

where $f(x)$ is a polynomial and k is a number. We can also have polynomial inequalities if we replace $\geq$ by $\leq$, $<$, or $>$.

Our goal when dealing with polynomial inequalities is to find the values of the domain that satisfy the inequality. In other words, to find the values of x that make the inequality true.

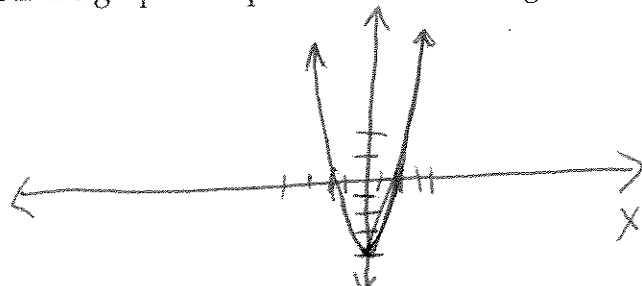
For example, if we're given the inequality:

$$x^2 \leq 4$$

then we can see that this is the same as the inequality:

$$x^2 - 4 \leq 0,$$

and if we graph the quadratic $x^2 - 4$ we get:



and so we can see that the values of x that satisfy our inequality are $-2 \leq x \leq 2$.

In general the way you figure out the values that satisfy a polynomial inequality is that you follow the steps:

- Move everything to one side! So, have a zero on one side, and everything else on the other.
- Use the fact that a polynomial can only change signs if it goes through a zero. So, calculate the roots of the polynomial.
- Check points in-between the roots to see if they're negative or positive. Reason accordingly.

Examples

1. Solve the inequality:

$$2x^2 + 5x \geq 12.$$

$$2x^2 + 5x - 12 \geq 0$$

$$2x^2 + 8x - 3x - 12$$

$$\Rightarrow 2x(x+4) - 3(x+4)$$

$$= (2x-3)(x+4)$$

$$\text{roots: } x = \frac{3}{2}, -4$$

$$2(0)^2 + 5(0) - 12 \geq 0$$

$$\Rightarrow -12 \geq 0 \text{ False.}$$

So,

$$\boxed{\begin{array}{c} x \leq -4 \\ \text{or} \\ x \geq \frac{3}{2} \end{array}}$$

2. Solve the inequality:

$$y^2 - 5y + 6 > 0.$$

$$(y-3)(y-2) > 0$$

$$\text{roots: } y = 2, 3$$

$$\text{check } y = \frac{5}{2}$$

$$\frac{25}{4} - \frac{25}{2} + 6 = \frac{25}{4} - \frac{50}{4} + \frac{24}{4} = -\frac{1}{4}.$$

So,

$$\boxed{y > 3 \text{ or } y < 2}$$

3. Solve the inequality:

$$x^2 + 2x + 1 \leq 0.$$

$$(x+1)^2 \leq 0$$

True for all x .

If $x^2 + 2x + 1 < 0$ true for all $x \neq -1$.

Finally, the last thing we'll look at are rational inequalities. So, inequalities of the form:

$$R(x) \geq k$$

or variants on this theme, where $R(x)$ is a rational function. To solve these we use exactly the same steps, except we note that a rational function can change its sign at the zeros (the zeros of the numerator), or at the points upon which it is undefined (the zeros of the denominator).

Example

Solve the inequality:

$$\frac{x+4}{x-2} > 0$$

Crit points: $x = 2, -4$.

Check: $x = 0$ $\frac{4}{-2} = -2 \not> 0$

Check: $x = 3$ $\frac{7}{1} = 7 > 0 \checkmark$

check: $x = -5$ $\frac{-1}{-7} = \frac{1}{7} > 0 \checkmark$

So,

$$\begin{array}{c} x < -4 \\ \text{or} \\ x > 2 \end{array}$$