Math 1010 - Lecture 26 Notes

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In today's lecture, I'll talk about how we find roots of quadratic equations. Basically, a quadratic equation is an equation of the form:

$$ax^2 + bx + c,$$

and finding the roots of this quadratic equation means finding the solutions to the equation:

$$ax^2 + bx + c = 0.$$

I say roots, plural, in that frequently[1] we'll have two solutions.

Just an example of a place where we've seen this before. Suppose we wanted to solve the equation:

$$x^2 - 3x + 2 = 0.$$

To find the roots of this, we'd want to factor the polynomial. Doing this we get:

$$(x-2)(x-1) = 0,$$

and we can see immediately that the roots will be $x = 1$ and $x = 2$.

[1] An in some sense, always unless $a = 0$.

1 Solving Quadratic Equations by Factoring and Special Forms

The example above illustrates a technique that we can use for solving quadratic equations if they can be easily factored. Just factor them, and find the roots of the two factors.

Examples

Solve the following equations:

1. $x^2 - 4x = -4$

$$x^2 - 4x + 4 = 0$$

so

$$(x-2)(x-2) = 0$$

from which we get $x = 2$.

2. $4x^2 - 12x + 9 = 0$

Factoring this we get:

$$(2x-3)(2x-3) = 0$$

and so

$$x = \frac{3}{2}.$$

Another example of an equation we already know how to solve is:

$$x^2 = 16.$$

The solutions here are just $x = 4$ and $x = -4$. In general if you have an equation in the form:

$$x^2 = d$$

the solutions are $x = \sqrt{d}$ and $x = -\sqrt{d}$.

Examples

Solve the following equations:

1. $x^2 = -3$

$$x = \pm\sqrt{-3} = \pm\sqrt{3}i.$$

2. $3x^2 = 15$

$$x^2 = 5;$$

so,

$$x = \pm\sqrt{5}.$$

The first example above illustrates that we can have imaginary solutions to quadratic equations. We can also have complex solutions. For example:

$$(x - 2)^2 = -9$$

has solutions $2 + 3i$ and $2 - 3i$.

We'll also sometimes be given equations that aren't quadratic, but are in "quadratic form", and are therefore susceptible to the same methods of attack as quadratic equations. For example, the equation:

$$x^4 - 3x^2 + 2 = 0$$

is not a quadratic equation, but it's in quadratic form, with x replaced by x^2. In fact, we basically did this problem at the beginning. We can factor it as:

$$(x^2 - 2)(x^2 - 1) = 0$$

and from this get the solutions $x^2 = 2$ and $x^2 = 1$. From these we get the solutions $x = \pm\sqrt{2}$ and $x = \pm 1$. So, there are four solutions to this equation.

2 Completing the Square

Completing the square comes from a desire to write our quadratic equation in the form:

$$a(x - b)^2 + c = 0.$$

The reason we want to write our equation this way is that in this format it's easy to solve. The solution will just be:

$$x = b \pm \sqrt{-\frac{c}{a}}.$$

How do we do this? Well, suppose we have the equation:

$$x^2 - 6x + 7 = 0.$$

Then to solve this we first subtract 7 from both sides:

$$x^2 - 6x = -7.$$

Then we want to write the left-hand side as a perfect square. So, we want to write the right hand side as:

$$(x+k)^2 = x^2 + 2kx + k^2.$$

Now, it's clear what k would have to be. If $2k = -6$ then $k = -3$. However, we don't have a $(-3)^2 = 9$ term on the left hand side. So, to get one, we add 9 to both sides:

$$x^2 - 6x + 9 = 2.$$

Then, we can write this as:

$$(x-3)^2 = 2.$$

Now, it's easy to solve it from here. Our solution will just be:

$$x = 3 \pm \sqrt{2}.$$

Examples

Solve the equation by completing the square.

1. $x^2 + 6x = 7$

If we add 9 to both sides we get:

$$x^2 + 6x + 9 = 16.$$

Now, $x^2 + 6x + 9$ is a perfect square: $(x+3)^2$.

So,

$$(x+3)^2 = 16,$$

and therefore

$$x = -3 \pm 4 = -7, 1.$$

2. $x^2 - 4x = -9$

If we add 4 to both sides we get:

$$x^2 - 4x + 4 = -5;$$

which means

$$(x-2)^2 = -5,$$

and so,

$$x = 2 \pm \sqrt{5}i.$$

3. $2x^2 + 8x + 3 = 0$

First, we can write this as:

$$2(x^2 + 4x) + 3 = 0.$$

Then, we can get our constants on the right hand side:

$$x^2 + 4x = -\frac{3}{2}.$$

Finally, to complete the square we add 4 to both sides:

$$x^2 + 4x + 4 = \frac{1}{2}.$$

Then, we write the left hand side as $(x+2)^2$ and get:

$$(x+2)^2 = \frac{1}{2}.$$

The solution then is:

$$x = -2 \pm \sqrt{\frac{1}{2}}.$$

3 The Quadratic Formula

Now let's look at this situation more generally. Suppose we have a quadratic equation:

$$ax^2 + bx + c = 0.$$

We want to solve this by completing the square. So, we can write this as:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

We can write the left hand side as a perfect square if we add $\dfrac{b^2}{4a^2}$ to both sides:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= -\frac{c}{a} + \frac{b^2}{4a^2} \\ \Rightarrow \left(x + \frac{b}{2a}\right)^2 &= -\frac{c}{a} + \frac{b^2}{4a^2}. \end{aligned}$$

Now, the right hand side we can rewrite as:

$$\frac{b^2 - 4ac}{4a^2}$$

So, if we solve the equation for x we get:

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

This simplifies to:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

or in other words:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is the famous **quadratic formula**! It gives you the solution to *any* quadratic equation. The only caveat here is that we assumed that $a > 0$, but it's always possible to set it up that way. If we have an equation:

$$ax^2 + bx + c = 0,$$

then if $a < 0$ we can multiply both sides by -1 to make $a > 0$.

Examples

Solve the following quadratic equations using the quadratic formula:

1. $x^2 - 2x - 6 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)} = \frac{2 \pm \sqrt{28}}{2} = 1 \pm \sqrt{7}.$$

2. $2x^2 - 2x + 3 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(3)}}{2(2)} = \frac{2 \pm \sqrt{-20}}{4} = \frac{1 \pm \sqrt{5}i}{2}.$$

3. $-15x^2 - 10x + 25 = 0$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-15)(25)}}{2(-15)}$$

$$= \frac{10 \pm \sqrt{100 + 1500}}{-30} = \frac{10 \pm 40}{-30} = 1, -\frac{5}{3}.$$