

Math 1010 - Lecture 24 Notes

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Today we're going to talk about how our standard arithmetic operations like addition, subtraction, multiplication, and division apply to radicals, and then we'll talk about how we can solve equations involving radicals.

1 Adding and Subtracting Radical Expressions

Two or more radical expressions are called *like radicals* if they have the same index and the same radicand. So, for example, $\sqrt{3}$ and $5\sqrt{3}$ are like radicals, while $\sqrt[3]{3}$ and $3\sqrt{3}$ are not. If we have like radicals, then we can combine them. So, for example:

$$\sqrt{3} + 4\sqrt{3} = 5\sqrt{3},$$

and

$$\sqrt[4]{5} - 3\sqrt[4]{5} = -2\sqrt[4]{5}.$$

Now, before we combine terms, we want to simplify. We'll frequently find that by simplifying our terms, things that appeared at first sight to not be like radicals actually are.

For example:

$$9\sqrt{50} - 4\sqrt{2} = 9(5\sqrt{2}) - 4\sqrt{2} = 41\sqrt{2}.$$

and

$$10\sqrt[3]{z} - \sqrt[3]{z^4} = 10\sqrt[3]{z} - z\sqrt[3]{z} = (10 - z)\sqrt[3]{z}.$$

Let's do some more examples:

Examples

Combine the radical expressions, if possible.

1. $13\sqrt{x} - \sqrt{x}$.

$$13\sqrt{x} - \sqrt{x} = \boxed{12\sqrt{x}}$$

2. $5\sqrt{12} + 16\sqrt{27}$.

$$\begin{aligned} & 5\sqrt{12} + 16\sqrt{27} \\ &= 10\sqrt{3} + 48\sqrt{3} = \boxed{58\sqrt{3}} \end{aligned}$$

3. $8\sqrt[4]{5} - 2\sqrt[3]{5}$.

cannot simplify

2 Multiplying and Dividing Radical Expressions

To multiply radical expressions, we use the product rule for radicals, which we've seen before. It's this rule:

$$\sqrt[n]{uv} = \sqrt[n]{u}\sqrt[n]{v}.$$

Examples

$$1. \sqrt{5}\sqrt{10} = \sqrt{50} = 5\sqrt{2}$$

$$2. \sqrt{10}(\sqrt{5} + \sqrt{6})$$

$$= \sqrt{50} + \sqrt{60} = 5\sqrt{2} + 2\sqrt{15}$$

$$3. (\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{3}) \quad \text{FOIL}$$

$$= \sqrt{5}\sqrt{5} - \sqrt{3}\sqrt{5} - \sqrt{3}\sqrt{5} + \sqrt{3}\sqrt{3}$$

$$= 5 - 2\sqrt{15} + 3 = \boxed{8 - 2\sqrt{15}}$$

Let's say we have an expression involving a radical like $2 - \sqrt{3}$. We say that the expression $2 + \sqrt{3}$ is its *conjugate*, and that the numbers $2 - \sqrt{3}$ and $2 + \sqrt{3}$ are conjugate. They differ only in the sign between the terms. Now, the product of two conjugate is the difference of squares:

$$(a - b)(a + b) = a^2 - b^2,$$

and this fact can be very useful when dealing with radicals, in that it can be used to get rid of the radicals!

For example:

$$(2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1.$$

Now, of course we can't just arbitrarily multiply terms, but we can use this very useful fact when we're dividing radical expressions.

For example, if we've got the expression:

$$\frac{\sqrt{2}}{2 - \sqrt{3}}$$

We can multiply the top and bottom of this equation by the conjugate of the denominator to eliminate the radical in the denominator:

$$\left(\frac{\sqrt{2}}{2-\sqrt{3}}\right)\left(\frac{\sqrt{2}}{2+\sqrt{3}}\right) = \frac{\sqrt{2}(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})} = \frac{2\sqrt{2}+\sqrt{6}}{1} = 2\sqrt{2}+\sqrt{6}.$$

Examples:

Simplify the following.

1. $\frac{4}{3\sqrt{5}-1}$

$$\frac{4}{3\sqrt{5}-1} \left(\frac{3\sqrt{5}+1}{3\sqrt{5}+1} \right) = \frac{12\sqrt{5}+4}{9(5)-1} = \frac{12\sqrt{5}+4}{44}$$

$$= \boxed{\frac{3\sqrt{5}+1}{11}}$$

2. $\frac{3x}{\sqrt{15}-\sqrt{3}}$

$$\frac{3x}{\sqrt{15}-\sqrt{3}} \left(\frac{\sqrt{15}+\sqrt{3}}{\sqrt{15}+\sqrt{3}} \right) = \frac{3x(\sqrt{15}+\sqrt{3})}{12} = \frac{x(\sqrt{15}+\sqrt{3})}{4}$$

3 Solving Radical Equations

So far we've just dealt with radical expressions. Now, we'll move on to radical equations. A radical equation is, not surprisingly, an equation involving a radical. For example:

$$\sqrt{x} - 5 = 2.$$

Now, the basic truth we use for solving radical equations is that if $u = v$ then $u^n = v^n$ for any positive integer n .

Using this fact, we can solve the above equation by noting:

$$\begin{aligned}\sqrt{x} &= 7 \\ \Rightarrow (\sqrt{x})^2 &= 7^2 \\ \Rightarrow x &= 49.\end{aligned}$$

Now, it's always important to check our answers to these radical equations, because we may have *extraneous* solutions, which are solutions that don't work when we plug them back into our original equation.

In this case:

$$\sqrt{49} - 5 = 2$$

is indeed true, so our solution $x = 49$ is correct.

Examples:

Solve the following radical equations:

1. $\sqrt{8x} = 6$ $(\sqrt{8x})^2 = 6^2$ Check: $\sqrt{8(\frac{9}{2})} = \sqrt{36} = 6 \checkmark$

$$\Rightarrow 8x = 36$$

$$\Rightarrow \boxed{x = \frac{9}{2}}$$

2. $\sqrt{3y+1} = 4$ $(\sqrt{3y+1})^2 = 4^2 \Rightarrow 3y+1 = 16 \Rightarrow 3y = 15 \Rightarrow \boxed{y = 5}$ Check: $\sqrt{3(5)+1} = \sqrt{16} = 4 \checkmark$

3. $\sqrt{9-2x} = -9$ $(\sqrt{9-2x})^2 = (-9)^2 \Rightarrow 9-2x = 81 \Rightarrow 2x = -72 \Rightarrow x = -36$

$$\sqrt{9-2(-36)} = \sqrt{81} = 9 \neq -9$$

5 $\boxed{\text{No solution}}$

Now, sometime we won't be able to isolate a radical on just one side, and sometimes we'll have to exponentiate more than once. The trick is to just get a radical on one side, and everything else to the other side, then exponentiate, and then if you've still got radicals, repeat the process.

Examples:

Solve for x :

1. $\sqrt{x+3} = \sqrt{2x-1}$.

$$(\sqrt{x+3})^2 = (\sqrt{2x-1})^2$$

$$\Rightarrow x+3 = 2x-1$$

$$\Rightarrow \boxed{4 = x}$$

check

$$\sqrt{4+3} = \sqrt{7}$$

$$\sqrt{2(4)-1} = \sqrt{7} \quad \checkmark$$

2. $\sqrt{2t+3} = 3 - \sqrt{2t}$.

$$(\sqrt{2t+3})^2 = (3 - \sqrt{2t})^2$$

$$2t+3 = 9 - 6\sqrt{2t} + 2t$$

$$\Rightarrow 6\sqrt{2t} = 6$$

$$\Rightarrow \sqrt{2t} = 1$$

$$\Rightarrow (\sqrt{2t})^2 = 1^2$$

$$\Rightarrow 2t = 1 = \boxed{t = \frac{1}{2}}$$

check:

$$\sqrt{2(\frac{1}{2})+3} = \sqrt{4} = 2$$

$$3 - \sqrt{2(\frac{1}{2})} = 3 - \sqrt{1} = 3 - 1 = 2$$

$\checkmark$