

Math 1010 - Lecture 18 Notes

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In our last lecture we talked about how we can add, subtract, and multiply polynomials, and we figured out that, basically, if you can add, subtract, and multiply numbers, you can do the same for polynomials. We postponed division of polynomials for a while because it's more difficult. Well, today we'll begin our investigation of how we divide polynomials by investigating how we can break up a complex polynomials into its factors, its factors being simpler polynomials that, when we multiply them all together, give us our complex polynomial.

An analogy with integers is perhaps in order. If we have an integer, say 24, we can frequently write it as a product of smaller numbers:

$$24 = 8 \times 3.$$

If we continue in this way, we can keep factoring our integer until we write it as a product of *prime numbers*, which are numbers that cannot be factored.¹ In our example this complete or prime factorization would be:

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3.$$

Today, we'll see how similar ideas apply to polynomials.

¹More precisely, numbers whose only factors are 1 and the number itself.

1 Common Monomial Factors

If we have a set of monomials, say $30x^5$, $15x^3$, $5x^4$, we say the greatest common factor of these monomials is the “largest” monomial that divides all three. By largest we mean the monomial of highest degree with the greatest coefficient. This greatest common factor will have a coefficient that is the greatest common factor of the coefficients or our monomials, and a degree that is the minimum degree of our monomials.

In our example the greatest common factor of our coefficients is 5, that being the greatest number that divides 30, 15 and 5, while the minimum of our degrees would be 3. So, the greatest common factor of our monomials is $5x^3$.

This idea is useful if you’ve got a polynomials with many monomial terms, and you want to *factor* out a monomials term. What this means is you want to rewrite your polynomial as a product of a monomial and a simpler polynomial. For example, if we take the polynomial:

$$4x^4 + 12x^3 - 8x^2$$

we can find the greatest common factor of our terms, in this case $4x^2$, and use this to rewrite our polynomial as:

$$4x^2(x^2 + 3x - 2).$$

Examples

Factor our the greatest common factor of the terms in the following polynomials.

1. $9x + 30$

$$3(3x + 10)$$

2. $x^2 + 9x$

$$x(x + 9)$$

3. $16 - 3y^3$

No common factor other than 1.

2 Factoring by Grouping and Special Forms

For some polynomials we might not be able to factor them much just by factoring out a monomial term, but we can factor them by grouping terms together, factoring those terms, and then combining those factorizations. For example, if we're given the polynomial:

$$t^3 - 11t^2 + t - 11$$

we can group the first two terms and the last two terms together, and then factor those, giving us:

$$(t^3 - 11t^2) + (t - 11) = t^2(t - 11) + (t - 11)$$

which, given our like terms, we can write as

$$(t^2 + 1)(t - 11).$$

Polynomials for which you can do this, that is for which once you group them and factor them you have like terms that you can combine, are relatively uncommon. However, it's good to get some practice at it so that when you see them you'll be able to spot them.

Exercises

Factor the following polynomials by grouping.

1. $y^2 + 3y + 4y + 12$.

$$\begin{aligned} & (y^2 + 3y) + (4y + 12) \\ = & y(y + 3) + 4(y + 3) = \boxed{(y + 4)(y + 3)} \end{aligned}$$

2. $3s^3 + 6s^2 + 5s + 10$.

$$\begin{aligned} & (3s^3 + 6s^2) + (5s + 10) \\ = & 3s^2(s + 2) + 5(s + 2) = \boxed{(3s^2 + 5)(s + 2)} \end{aligned}$$

Now, there are certain special forms of polynomials that you should learn how to spot and factor right away. Namely, a difference of squares, and sums and differences of cubes.

- $x^2 - y^2 = (x + y)(x - y)$
- $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
- $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Examples

Factor the following polynomials.

1. $16y^2 - 9$.

$$(4y + 3)(4y - 3)$$

2. $m^3 - 8n^3$.

$$(m - 2n)(m^2 + 2nm + 4n^2)$$

3 Complete Factorization

Finally, we note that to factor a polynomial completely is to write it as a product of terms that cannot be factored further, at least with integer coefficients. This is analogous to writing a number as the product of primes.

Examples

Factor the following polynomials completely.

1. $y^4 - 81$

$$(y^2 - 9)(y^2 + 9)$$

$$= \boxed{(y - 3)(y + 3)(y^2 + 9)}$$

2. $3x^4 - 300x^2$

$$3x^2(x^2 - 100) = \boxed{3x^2(x - 10)(x + 10)}$$

4 Factoring Trinomials

A particular type of polynomial that we'll be factoring quite a bit is a trinomial in the form:

$$ax^2 + bx + c$$

For example, we may want to factor the trinomial:

$$x^2 + 6x + 9.$$

The first thing we do is check if there are any common factors of the terms other than 1, and if there are, we factor out the largest one. In this case there are not. The next thing we do is try to write our trinomial as the product of two binomials. In other words, we want to write it as:

$$x^2 + 6x + 9 = (ax + b)(cx + d).$$

In practice, what we do here is we look at the factors of the coefficient for x^2 , and make those our possible values of a and c . We do the same thing for the constant term 9, and make those our possible values of b and d . Then, we just check the possible combinations to see if any of them work out. More precisely, we have:

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$$

and we want to find factors ac and bd such that, in our example, $ad + bc = 6$. The only factors of 1 are $a = 1, c = 1$ and $a = -1, c = -1$. The only factors of 9 are $b = 3, d = 3$ and $b = -3, d = -3$. If we try the positive 1 terms and the negative 3 terms we would get:

$$(x - 3)(x - 3) = x^2 - 6x + 9 \neq x^2 + 6x + 9.$$

So, that doesn't work. Let's try when they're both positive.

$$(x + 3)(x + 3) = x^2 + 6x + 9.$$

So, we have $x^2 + 6x + 9 = (x + 3)^2$. In practice, this is what we do. We look at the factors of the constant term and the coefficient of x^2 , and try different combinations until we find one that works. Note that we're not guaranteed to find one that works. In practice these become easier and easier as you do more of them and start learning the tricks and the patterns. For example, if the constant term is much greater than the coefficient of x , you look for factors of the constant term that are close in absolute value, so their sum or difference will be small.

Examples

Factor the following trinomials.

1. $25y^2 - 10y + 1$

$$(5y - 1)(5y - 1) = (5y - 1)^2$$

2. $x^2 + 2x - 8$

$$(x + 4)(x - 2)$$

3. $x^2 + 7x + 10$

$$(x + 5)(x + 2)$$

4. $5z^2 + 2z - 3$

$$(5z - 3)(z + 1)$$

5. $6x^2 - 5x + 6$

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Prime