Math 1010 - Lecture 13 Notes

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In today's lecture we will discuss relations and functions, and go over the important definitions and concepts associated with them, such as domain and range. We will then discuss how we can graph functions from real numbers to real numbers, and how these graphs can help us understand functions.

1 Relations and Functions

A relation is any set of ordered pairs. The *domain* of the relation is the set of first components, while the *range* of the relation is the set of second componends. For example, the following set of pairs would be a relation:

$$\{(2,3),(2,5),(3,3),(1,4)\}.$$

The domain of this relation is the set:

$$\{1,2,3\},$$

while the range is the set:

$$\{3,4,5\}.$$

A *function* is a specific type of relation, in which every element in the domain is assigned one *and only one* element in the range. So, if you view the domain as the set of possible inputs, and the range as the set of possible outputs, a function is a relation in which every input has only one output. Note that we could have many inputs going to the same output, but we can't have many outputs going to the same input.

Examples

1. Would the relation that takes a person and outputs the person's maternal grandmother be a function?

   Yes. Every person has one and only one maternal grandmother.

2. Would the relation that takes a person and outputs the person's siblings (if any) be a function?

   No. A person could have no siblings or more than one.

3. Would the relation that takes a real number and squares it be a function?

   Yes. Every real number has a unique square.

4. Would the relation that takes a real number and finds its square root be a function?

   No. For example $\sqrt{4} = \pm 2$. So, one input can map to more than one output.

Now, when we have a discrete (finite) set of inputs and a discrete set of outputs we can frequently represent a relation or a function as just a list of ordered pairs. However, when we've got larger domains (such as all real numbers) a list of ordered pairs becomes unreasonable, or even impossible. In these cases we frequently use an equation to represent a relation of function. For example, the equation:

$$y = x^2$$

could be viewed as a function. It takes a set of inputs (x-values) and for any given input it gives us a specific output (y-value). So, for example, if we input $x = 3$ into this equation, we'd get an output $y = 3^2 = 9$. In an equation we call the input variable the *independent variable*, and the output variable the *dependent variable*. In the above eqution x is the independent variable, while y is the dependent variable.

When we have an equation representing a function, it's commonly convenient to give our function a name so that it can be easily referenced. Frequently, we call the function "f", and write our equation:

$$f(x) = x^2 + 1$$

for example. What this means is that our domain is the possible x-values, the inputs, and for each input x our f takes that input and calculates $f(x)$, the output. We read $f(x)$ as the value of f at x, or as simply f of x.

If we want to take a particular input, say $x = 3$, and calculate the corresponding particular output we write:

$$f(3) = 3^2 + 1 = 10.$$

This means that for an input of 3 our output $f(3)$ is 10.

Examples

For the eqution $g(x) = x^2 + 3$ calculate:

1. $g(3)$.

$$g(3) = 3^2 + 3 = 9 + 3 = \boxed{12}$$

2. $g(-2)$.

$$g(-2) = (-2)^2 + 3 = 4 + 3 = \boxed{7}$$

3. $g(x+1)$.

$$g(x+1) = (x+1)^2 + 3 = \boxed{x^2 + 2x + 4}$$

Finally, if we're just given an equation, we take the domain to be all real numbers for which the equation makes sense. In other words, all real numbers where the equation is defined. For example, the equation:

$$f(x) = \frac{x-3}{x+1}$$

has a domain of all real numbers except $x = -1$. This is because our function is defined for all real numbers except $x = -1$. At $x = -1$ we have division by zero, which does not make sense and is undefined.

2 Graphs of Functions

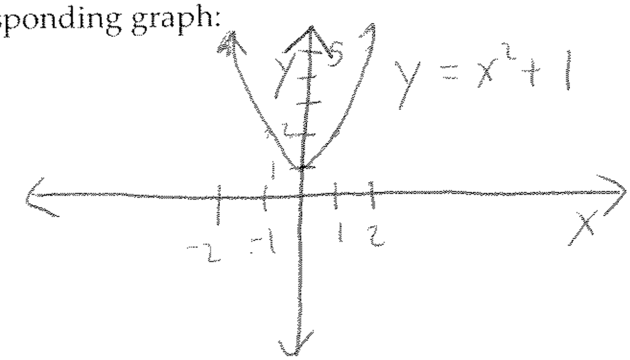
If we're given a function f whose domain and range are sets of real numbers, we can construct a graph of the function. We represent inputs on the x-axis, and corresponding outputs on the y-axis. For example, the function:

$$f(x) = x^2 + 1$$

we'd represent as the equation:

$$y = x^2 + 1$$

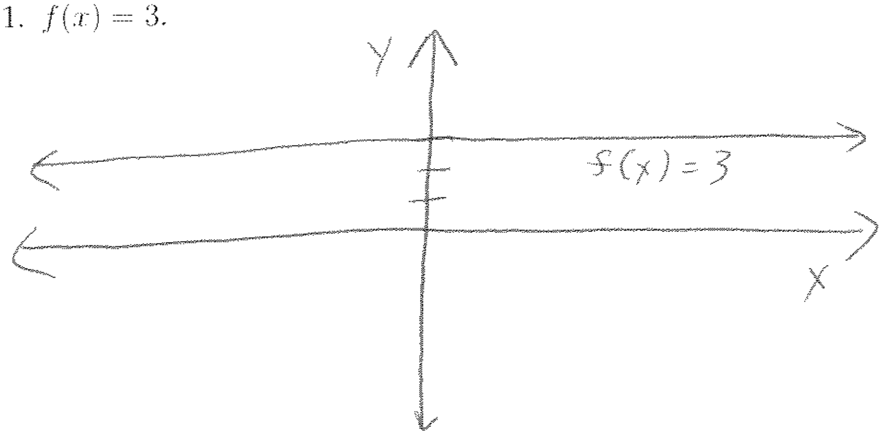
with corresponding graph:



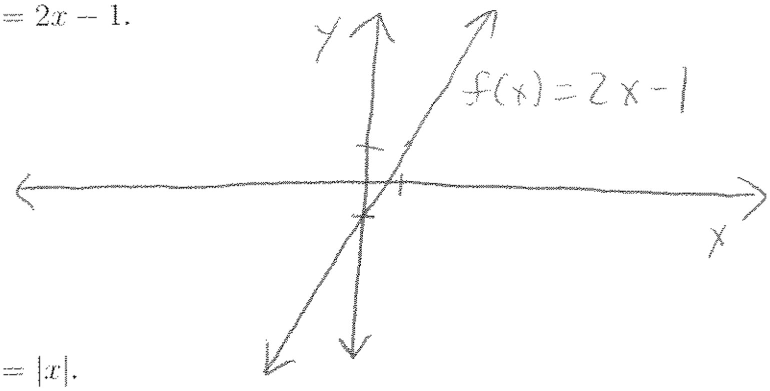
Examples

Sketch the graphs of the following functions:

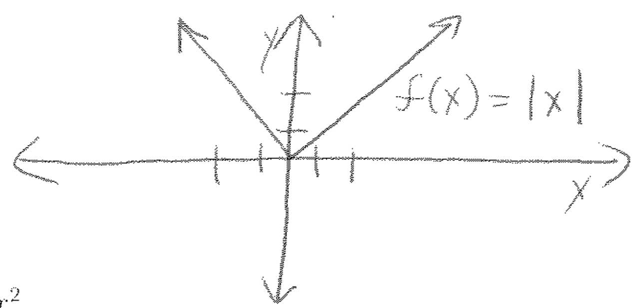
1. $f(x) = 3$.



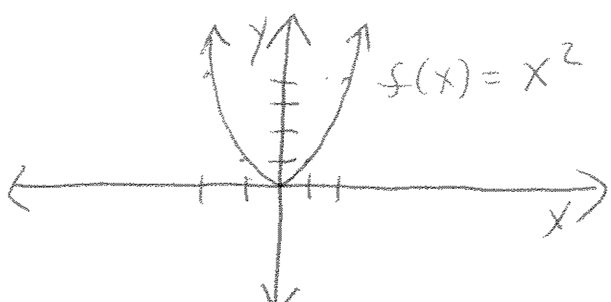
2. $f(x) = 2x - 1$.



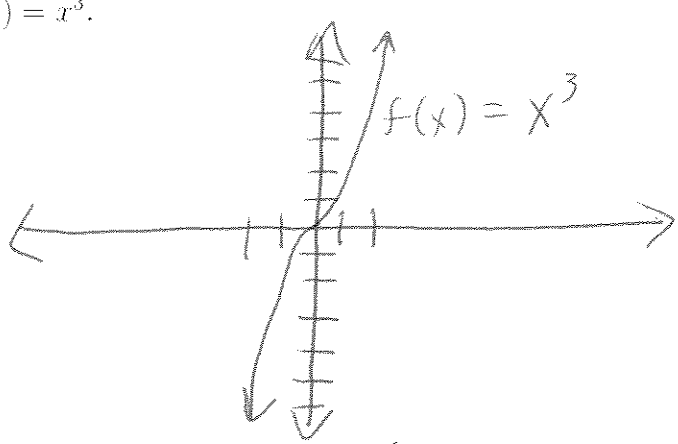
3. $f(x) = |x|$.



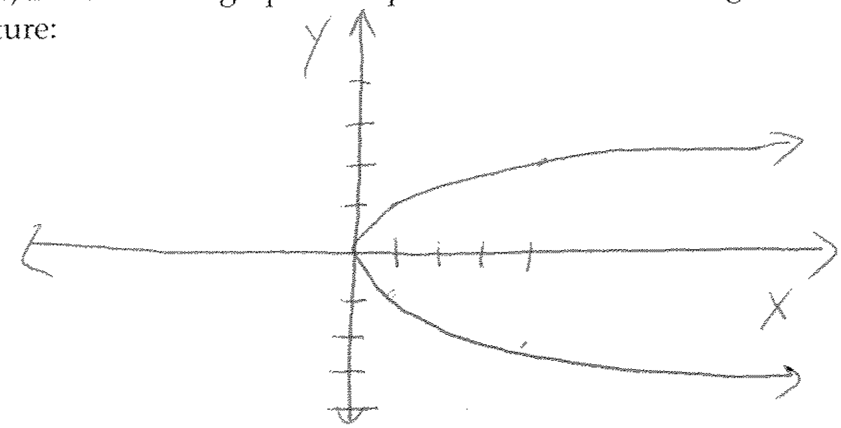
4. $f(x) = x^2$.



5. $f(x) = x^3$.



Now, if we were to graph the square root relation we'd get the following picture:



As mentioned before, this relation isn't a function, because for a given input, say 4 we could get more than one output, in this case 2 and -2. If we have the graph of a relation and we want to check if it's a function, we can use the *vertical line test*. The vertical line test is simply that if we can draw a vertical line anywhere on our picture that hits our graph more than once, then this corresponds with two possible outputs for a given input, and therefore our graph does not represent a function. The square root relation drawn above fails the vertical line test, but all the examples we drew previously pass the vertical line test, and therefore we can tell from the graphs that they represent functions.

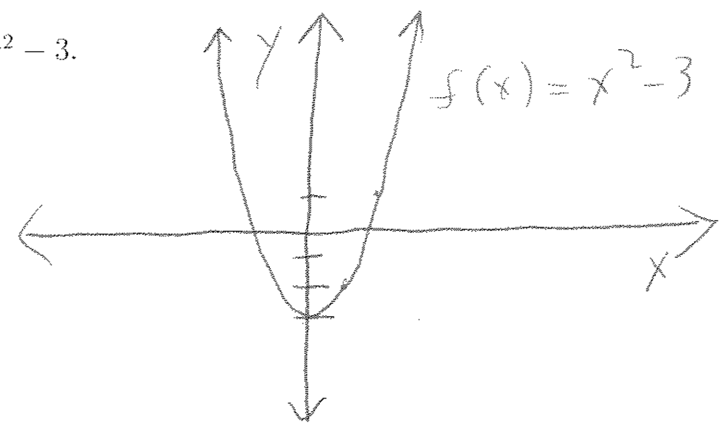
Given a function $f(x)$, we can shift it either vertically or horizontally according to the following rules:

- Vertical shift c units *upwards* : $f(x) + c$.
- Vertical shift c units *downwards* : $f(x) - c$.
- Horizontal shift c units to the *right* : $f(x - c)$.
- Horizontal shift c units to the *left* : $f(x + c)$.

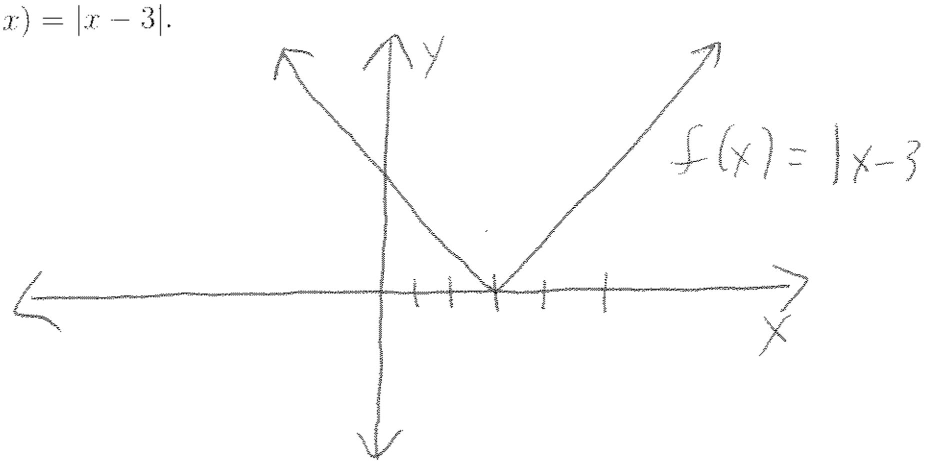
Examples

Graph the following functions:

1. $f(x) = x^2 - 3$.



2. $f(x) = |x - 3|$.



Finally, we can reflect a graph around either the x or y axis through the following transformations:

- Reflection around the x-axis corresponds to $f(x) \to -f(x)$.
- Reflection around the y-axis corresponds to $f(x) \to f(-x)$.

Example

1. Graph the function $f(x) = -x^2$.