Math 2210 - Section 14.1 Line Integrals

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1 Concept of a Line Integral

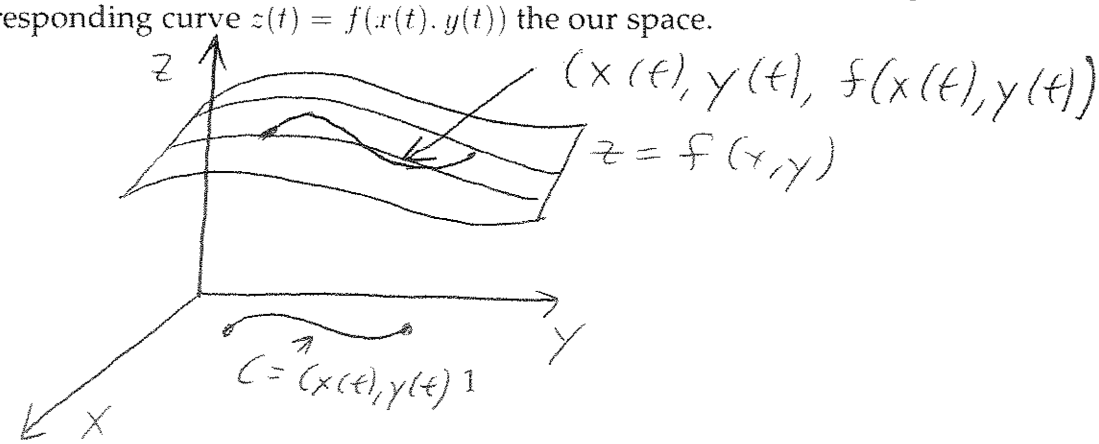
Way back when in chapter 11 we discussed parameterized curves, and we recall that a parameterized curve in 2-dimensions is a curve whose x and y coordinates are both functions of another parameter, usually denoted as t:

$$x = x(t)$$
$$y = y(t)$$

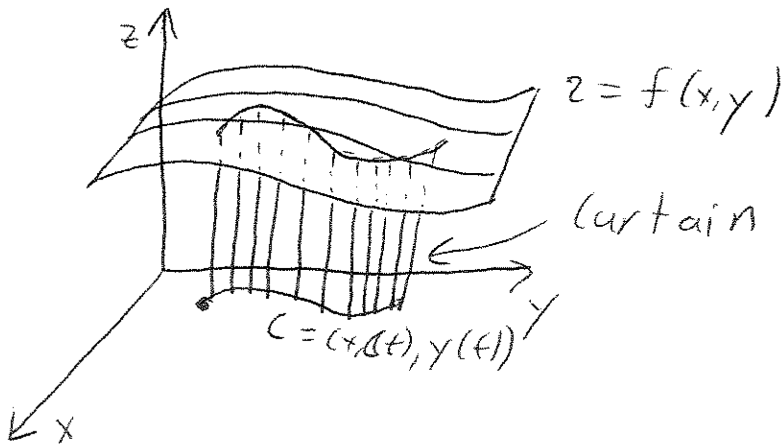
So, for example, we could parameterize a unit circle as:

$$x = \cos t$$
$$y = \sin t$$
$$0 \leq t < 2\pi.$$

Now, if we have a surface defined by a two-variable function $z = f(x, y)$ then if we take a curve in the xy-plane, this curve will map to a corresponding curve $z(t) = f(x(t), y(t))$ the our space.



Now, if over every point along this curve we draw a line projecting it to the xy-plane, our resulting shape is a two-dimensional "curtain" defined by the curve and the surface:



Now, how do we calculate the area of this curtain? Well, the curve occurs as our parameter t varies from some initial value a to some final value b. We can break up this interval into N segments of length Δt and then get an approximation for this area:

$$A = \sum_{k=1}^{N-1} f(x(a+k\Delta t), y(a+k\Delta t))\sqrt{\left(\frac{\Delta x_k}{\Delta t}\right)^2 + \left(\frac{\Delta y_k}{\Delta t}\right)^2}$$

$$\text{where } \Delta x_k = x(a+(k+1)\Delta t) - x(a+k\Delta t)$$
$$\text{and } \Delta y_k = y(a+(k+1)\Delta t) - y(a+k\Delta t).$$

Now, as you might expect, we take the limit as Δt goes to 0, and hence N goes to infinity, and our limit is called a line integral:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

We note here that our definition of the line integral is in fact defined in terms of the curve C, and not in terms of the paramaterization. This is because the curve C and the function $f(x,y)$ are what's really important about it, and if we take two different paramaterizations of the curve C, as long as both paramaterizations start and end at the same point we'll get the same value for the line integral.

We finally note that the same concept, if not the ability to picture it, extends to curves in three dimensional space and functions of three variables:

$$\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

I just note here at the end that there is much more we could talk about in relation to line integrals, but unfortunately we're out of time. Sorry we got so far behind this semester.

2 Examples

Example - Evaluate $\int_C xy^{\frac{2}{5}}ds$ where C is the curve given by $x = \frac{1}{2}t$, $y = t^{\frac{5}{2}}$, $t \in [0,1]$.

$$\int_C xy^{2/5}ds = \int_0^1 \left(\frac{1}{2}t\right)\left(t^{5/2}\right)^{2/5}\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}t^{3/2}\right)^2}\,dt$$

$$= \frac{1}{2}\int_0^1 t^2\sqrt{\frac{1}{4} + \frac{25}{4}t^3}\,dt$$

$$= \frac{1}{4}\int_0^1 t^2\sqrt{1+25t^3}\,dt \qquad \begin{aligned} u &= 1+25t^3 \\ du &= 75t^2dt \end{aligned}$$

$$= \frac{1}{300}\int_1^{26}\sqrt{u}\,du = \frac{1}{450}u^{3/2}\Big|_1^{26}$$

$$= \boxed{\frac{26\sqrt{26}-1}{450}}$$

Example - Evaluate $\int_C (x^2 + y^2 + z^2)ds$ where C is the curve given by $x = 4\cos t$, $y = 4\sin t$, and $z = 3t$, where $0 \leq t \leq 2\pi$.

$$\int_C (x^2+y^2+z^2)ds = \int_0^{2\pi} (16\cos^2 t + 16\sin^2 t + 9t^2)\sqrt{16\sin^2 t+16\cos^2 t+9}\, d$$

$$= 5\int_0^{2\pi} (16+9t^2)\,dt$$

$$= 5\left(16t+3t^3\right)\Big|_0^{2\pi}$$

$$= 5(32\pi + 24\pi^3)$$

$$= \boxed{160\pi + 120\pi^3}$$