Math 2210 - Section 14.1 Vector Fields

Dylan Zwick

Fall 2008

1 Concept of a Vector Field

So far, we've studied functions from one real variable to one real variable,

$$y = f(x),$$

functions of one real variable to multiple real variables,

$$\mathbf{r} = \mathbf{f}(t) =< f_1(t), f_2(t), \ldots, f_n(t) >,$$

(where n is the number of variables in our range), and functions of multiple variables to one variable,

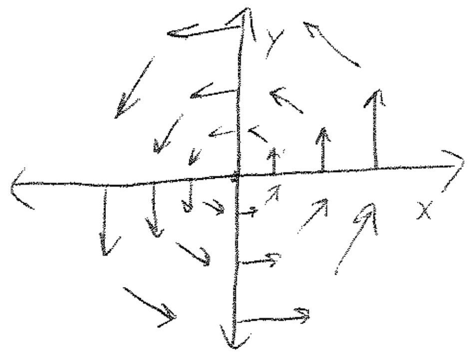
$$z = f(x, y) \text{ and } w = f(x, y, z).$$

The study of the first type (one variable to one variable) was the subject of calculus I and calculus II, and has formed the foundation for our study of the other types. The study of the second type was our subject in vector-valued functions, and only occupied us briefly in chapter 11, as it essentially just boiled down to single-variable calculus, done multiple times. The study of the third type (multiple variables to one variable) was a more difficult transition, and has formed the subject of most of this course. Well, I bet you can guess what our next subject is going to be. It will be the study of functions of multiple variables to multiple variables. Such relations are called *vector fields*.

A vector field (at least in lower dimensions) can be viewed as a function $\mathbf{F}$ that associates each vector input $\mathbf{p}$ (viewed as a point in n-space $(x_1, x_2, \ldots, x_n)$) with an output vector $\mathbf{F}(\mathbf{p})$. For example, a map from two variables to two variables would be:

$$\mathbf{F}(\mathbf{p}) = \mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

Now, we cannot really draw this, as we would have to draw an arrow at an uncountably infinite number of points, and the result would just be a big opaque mess. However, if we draw some representative vectors we can get an idea for what this vector fields "looks like":



Now, if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ (the vector that points from the origin to the point (x, y)) then if we take the dot product of this position vector with the vector from our vector field at the point (x, y) we get:

$$\mathbf{F}(x, y) \cdot \mathbf{r} = (-y\mathbf{i} + x\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j}) = -xy + xy = 0$$

which means that the position vector $\mathbf{r}$ is always perpendicular to the vector field $\mathbf{F}$. (Note that this analytical fact corresponds to our picture of the vector field). Also, we note that

$$||\mathbf{F}(x, y)|| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = ||\mathbf{r}||.$$

So, the magnitude of the vector from the vector field at point (x, y) is equal to the distance from the origin to the point (x, y). One thing this implies is that the vectors of our vector field are getting larger the farther away they are from the origin, which is also reflected in our earleir diagram.

2 Newton's Law of Universal Gravitation

Isaac Newton, a very smart man, deduced in his studies of astronomy that the orbits of the planets around the sun, as well as the trajectories of the comets, could be explained by postulating that between any two bodies there is an attractive force called gravity. Newton postulated (actually derived, as it can be derived from Kepler's laws of planetary motion and Newton's laws of motion) that this force would be directly proportional to the product of the masses of the two planets, and inversely proportional to the distance between the two planets. As a formula, he stated that:

$$||\mathbf{F}|| = \frac{GMm}{d^2}$$

where d is the distance between the two planets, M and m are the masses of the two planets respectively, and G is the proportionality constant, called Newton's gravitational constant, and equal to

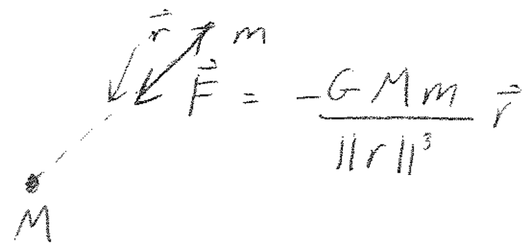
$$G \approx 6.67 \times 10^{-11} Nm^2/kg^2.$$

Example

If we view our larger mass (say the sun) as being a point mass centered at the origin then if we place another mass m at some position (x, y, z) the force experienced by the smaller mass will be directed directly towards the sun, and will have a magnitude given by Newton's law of universal gravitation. As a formula, we can express this as:

$$\mathbf{F} = -\frac{GMm}{||\mathbf{r}||^3}\mathbf{r}$$

where $\mathbf{r} = < x, y, z >$ is a vector that points from the origin to the point (x, y, z), and so lies along a line connecting the mass m with the central mass M.



3 Gradient and Curl

For three dimensional vector fields a general vector field can be expressed as:

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

Now, upon the vector field we have two common operations that we can perform. (Note that for all examples here we assume that the second partial derivatives of the scalar valued functions M, N and P exist and are continuous). These operations are the divergence of the vector field F, and the curl.

The divergence of the vector field is defined as:

$$div(\mathbf{F}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

While the curl is defined as:

$$curl(\mathbf{F}) = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

Note that the divergence of a vector field **F** returns a scalar valued function, while the curl of a vector field **F** returns another vector field. Now, these are the formal definitions, so in some sense that's all there is to it. However, the divergence and curl are frequently written in a very suggestive notation:

$$div(\mathbf{F}) = \bigtriangledown \cdot \mathbf{F}$$

and

$$curl(\mathbf{F}) = \bigtriangledown \times \mathbf{F}.$$

Now, why do we write these operations like this? Well, if we allow an abuse of notation we can call $\bigtriangledown$ the "dell" operator and define it as:

$$\bigtriangledown = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >.$$

If we continue in this fashion and, again abusing notation all over the place, if we treat the "dell"operator as a vector and our vector field **F** as another vector then we get:

$$\nabla \cdot \mathbf{F} = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} > \cdot < M, N, P > = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

and

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} =$$

$$\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

Which is the origin of the suggestive notation, and is also a useful way to remember how these operations are performed.

3.1 Interpretation

If we interpret the vector field **F** as representing a velocity field for a moving fluid, then the divergence of **F** at a point (x, y, z) represents the average amount of mass moving into a small area around the point. The curl is a little more difficult to understand. Around any axis through the point (x, y, z) there will be some angular velocity of the fluid. The curl picks out the maximum value over all these angular velocities. So, the curl points in the direction of maximal angular velocity, and the magnitude of the curl gives the angular speed of this angular velocity.

4 Example Problems

Example - For each of the following vector fields calculate the divergence and the curl.

1. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$\nabla \cdot \vec{F} = \boxed{2x + 2y + 2z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k} = \boxed{\vec{0}}$$

2. $\mathbf{F}(x, y, z) = \cos x\mathbf{i} + \sin y\mathbf{j} + 3\mathbf{k}$

$$\nabla \cdot \vec{F} = -\sin x + \cos y = \boxed{\cos y - \sin x}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & \sin y & 3 \end{vmatrix} = (0-0)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k} = \boxed{\vec{0}}$$

3. $\mathbf{F}(x, y, z) = (y + 2z)\mathbf{i} + (-x + 3z)\mathbf{j} + (x + y + z)\mathbf{k}$

$$\nabla \cdot \vec{F} = 0 + 0 + 1 = \boxed{1}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+2z & -x+3z & x+y+z \end{vmatrix} = (1-3)\hat{i} + (2-1)\hat{j} + (-1-1)\hat{k} = \boxed{-2\hat{i} + \hat{j} - 2\hat{k}}$$

Example - Show that $div(curl(\mathbf{F})) = 0$.

$$\vec{F}(x,y,z) = M\hat{i} + N\hat{j} + P\hat{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y}$$

Now, $\dfrac{\partial^2 P}{\partial x \partial y} = \dfrac{\partial^2 P}{\partial y \partial x}$, $\dfrac{\partial^2 N}{\partial x \partial z} = \dfrac{\partial^2 N}{\partial z \partial x}$, $\dfrac{\partial^2 M}{\partial y \partial z} = \dfrac{\partial^2 M}{\partial z \partial y}$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = 0 \quad \checkmark$$

Example - Show that $curl(grad(f)) = \mathbf{0}$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\hat{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}. \quad \checkmark$$