Math 2210 - Section 13.8 Triple Integrals in Cylindrical and Spherical Coordinates

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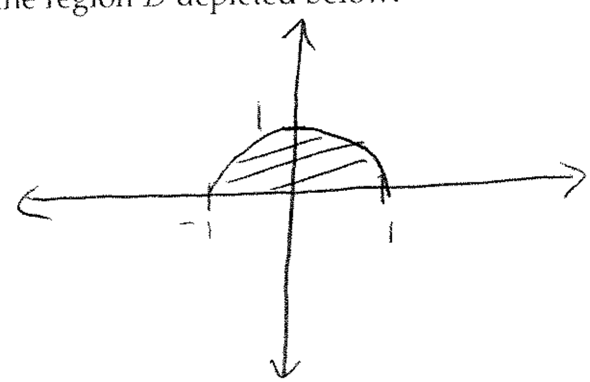
1 Cylindrical Coordinates

When we were dealing with double integrals, we found that with domains D that have a level of symmetry around a point, especially around the origin, it's frequently easier to calculate the double integrals by first converting to polar coordinates. For example, if we want to calculate the double integral of the function:

$$f(x, y) = x^2 + y^2$$

over the region D depicted below:



In Cartesian coordinates this integral would be:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$$

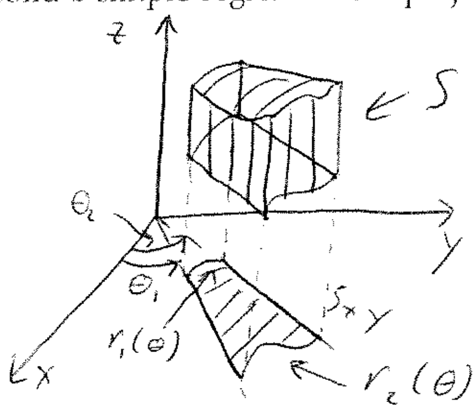
which, especially due to the $\sqrt{1-x^2}$ term in the inner integral, is very difficult to integrate. However, if we convert to polar coordinates we get the integral:

$$\int_0^{\pi}\int_0^1 r^3\,dr\,d\theta$$

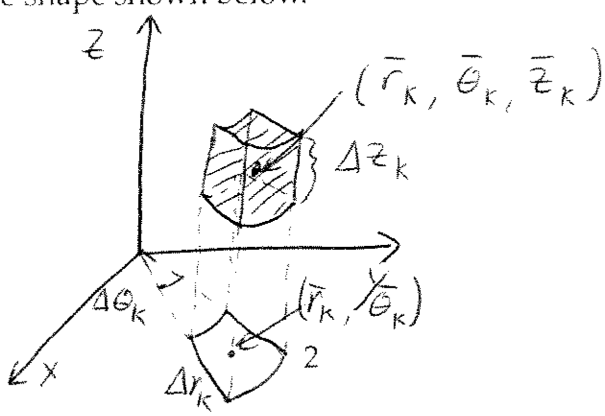
which is so easy to integrate we can pretty much do it in our heads.

Well, the same basic idea can be applied to three dimensional systems. Suppose we want to integrate a function $f(x, y, z)$ over a domain S, and the domain S displays a large amount of symmetry about an axis, especially the z-axis. In this case, it's frequently easier to do the triple integral by converting to cylindrical coordinates.

The basic idea is this. Suppose we wish to evaluate $\int\int\int_S f(x, y, z)dV$, where S is a solid z-simple region whose projection onto the xy-plane is r-simple.



Partition S by means of a cylindrical grid, where the typical volume element has the shape shown below.



This piece is called a cylindrical wedge, and has volume $\Delta V_k = \overline{r}_k \Delta z_k \Delta r_k \Delta \theta_k$. We can approximate the triple integral as:

$$\int\int\int_S \approx \sum_{k=1}^{n} f(\overline{r}_k \cos\overline{\theta}_k, \overline{r}_k \sin\overline{\theta}, \overline{z}_k)\overline{r}_k \Delta z_k \Delta r_k \Delta\theta_k$$

where we use the transformation maps for going from Cartesian to cylindrical:

$$x = r\cos\theta \; y = r\sin\theta \; z = z.$$

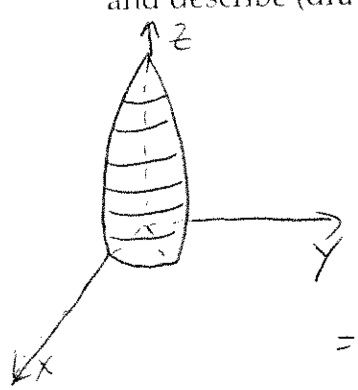
Now, if we take the limit as the norm of our partition goes to 0 (our usual strategy) we get:

$$\int\int\int_S f(x, y, z) = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta.$$

Example
Evaluate the integral:

$$\int_0^{\frac{\pi}{4}} \int_0^3 \int_0^{9-r^2} zr\,dz\,dr\,d\theta$$

and describe (draw) the region of integration.



$$\int_0^{\pi/4} \int_0^3 \int_0^{9-r^2} z r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^3 \frac{(9-r^2)^2}{2} r\,dr\,d\theta$$

$u = 9 - r^2$
$du = -2r\,dr$

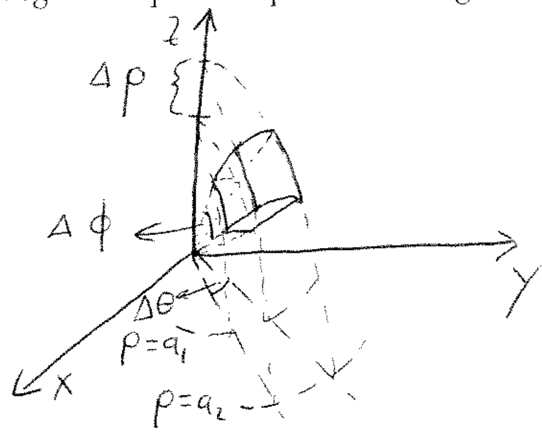
$$= \int_0^{\pi/4} \int_9^0 -\frac{u^2}{4}\,du\,d\theta$$

$$= \int_0^{\pi/4} \frac{u^3}{12}\Big|_0^9 d\theta = \int_0^{\pi/4} \frac{243}{4}\,d\theta = \boxed{\frac{243\pi}{16}}$$

2 Spherical Coordinates

Now, suppose we have a region of integration S that exhibits a large amount of symmetry around a point, especially the origin. Then this region is frequently easiest to describe in terms of spherical coordinates.

Now, the theory behind integration in spherical coordinates is basically the same as the theory behind integration in cylindrical coordinates. We partition our region S up into "spherical wedges" that look like the figure below.



The volume of this spherical wedge is:

$$\Delta V = \overline{\rho}^2 \sin\theta \Delta\rho\Delta\theta\Delta\phi$$

and so if we take the limit over our partitions as the norm approaches zero, we get the following equality:

$$\int\int\int_S f(x,y,z)dV =$$

$$\int_{\phi_1}^{\phi_2}\int_{\theta_1(\phi)}^{\theta_2(\phi)}\int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi.$$

Where we've used the transformations:

$$x = \rho\cos\theta\sin\phi, \; y = \rho\sin\theta\sin\phi, \; z = \rho\cos\phi.$$

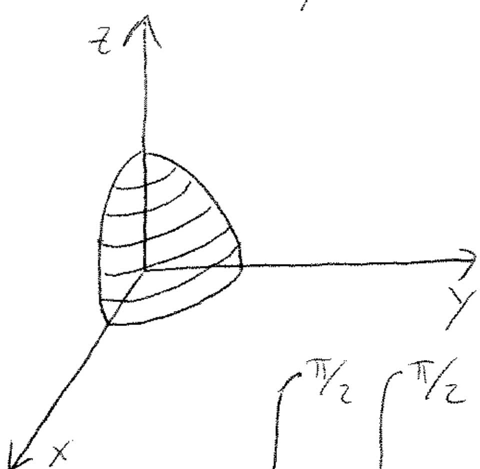
Example

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{a} \rho^2 \cos^2\phi \sin\phi d\rho d\theta d\phi$$

and describe (sketch) the region of integration R.

The region R looks like:



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \rho^2 \cos^2\phi \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^3}{3}\cos^2\phi \sin\phi \Big|_{\rho=0}^{\rho=a} d\theta \, d\phi$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\theta \, d\phi$$

$$= \frac{a^3}{3}\int_0^{\pi/2}\left(\cos^2\phi \sin\phi \, \theta\Big|_0^{\pi/2}\right) d\phi$$

$$= \frac{\pi a^3}{6}\int_0^{\pi/2} \cos^2\phi \sin\phi \, d\phi \qquad u = \cos\phi, \quad du = -\sin\phi \, d\phi$$

$$= \frac{\pi a^3}{6}\int_1^{0} -u^2 du = \boxed{\frac{\pi a^3}{18}}$$

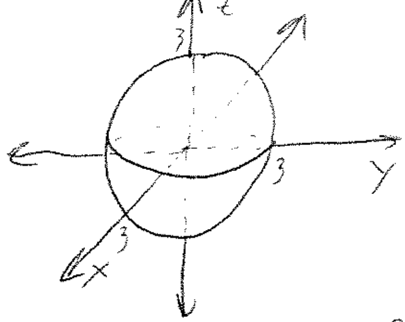
3 Examples

Example

Evaluate the integral:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}}(x^2+y^2+z^2)^{\frac{3}{2}}\,dydzdx$$

The set over which we're integrating is the ball of radius 3 centered at the origin.



So, converting to spherical coordinates we get:

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{3}\rho^5\sin\phi\,d\rho\,d\theta\,d\phi$$

$$=\int_0^{\pi}\int_0^{2\pi}\frac{\rho^6}{6}\sin\phi\Big|_{0=\rho}^{3=\rho}d\theta\,d\phi$$

$$=\int_0^{\pi}\int_0^{2\pi}\frac{243}{2}\sin\phi\,d\theta\,d\phi$$

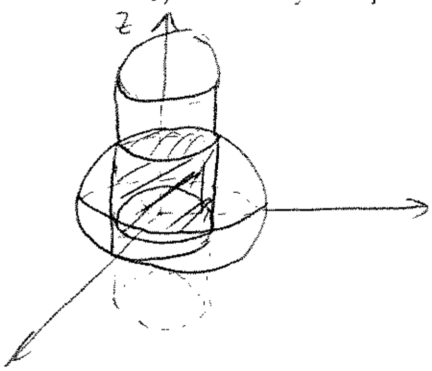
$$=\int_0^{\pi}\frac{243}{2}\sin\phi\;\theta\Big|_{\theta=0}^{\theta=2\pi}d\phi$$

$$=243\pi\int_0^{\pi}\sin\phi\,d\phi=-243\pi\cos\phi\Big|_0^{\pi}$$

$$=-243\pi(-1-1)=\boxed{486\pi}$$

Example

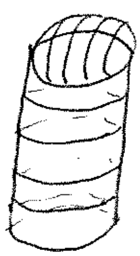
Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2=9$, below by the plane $z=0$, and laterally by the cylinder $x^2+y^2=4$.



We want to convert this to cylindrical:

$$\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r\sqrt{9-r^2}\,dr\,d\theta$$



Looks like this

$u = 9-r^2$

$du = -2r\,dr$

$$= \int_0^{2\pi}\int_9^{5} -\frac{1}{2}\sqrt{u}\,du\,d\theta$$

$$= \int_0^{2\pi}\int_5^{9} \frac{\sqrt{u}}{2}\,du\,d\theta = \int_0^{2\pi} \left.\frac{u^{3/2}}{3}\right|_5^9 d\theta$$

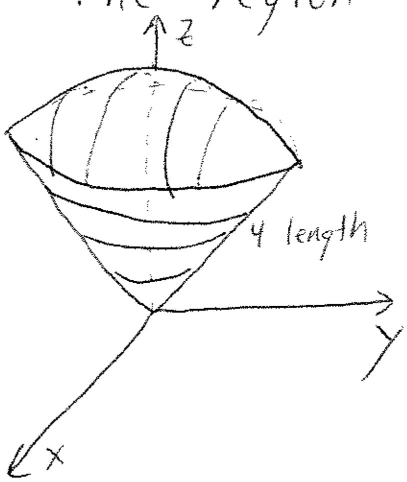
$$= \int_0^{2\pi}\left(9-\frac{5\sqrt{5}}{3}\right)d\theta$$

$$= \boxed{2\pi\left(9-\frac{5\sqrt{5}}{3}\right)}$$

Example

Find the volume of the solid within the sphere $x^2+y^2+z^2=16$, outside the cone $z=\sqrt{x^2+y^2}$ and above the xy-plane.

The region looks like a snowcone:



Converting this to spherical we get:

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{4} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{\rho=4} d\theta \, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \frac{64}{3}\sin\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi/4} \frac{128}{3}\pi \sin\phi \, d\phi$$

$$= \frac{128}{3}\pi \left(-\cos\phi \Big|_0^{\pi/4}\right)$$

$$= \frac{128}{3}\pi\left(-\frac{\sqrt{2}}{2}+1\right) = \boxed{\frac{64}{3}\pi\left(2-\sqrt{2}\right)}$$