Math 2210 - Section 13.7 Triple Integrals in Cartesian Coordinates

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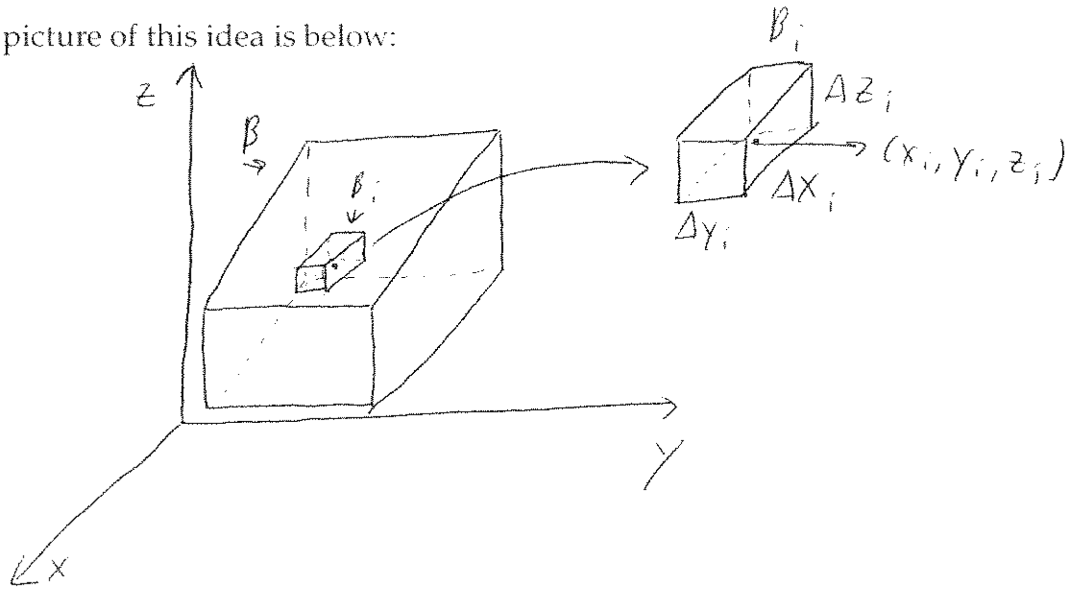
1 Concepts and Formulas

The basic idea behind a double integral can be naturally extended into triple integrals, or even to n-dimensional integrals. However, as we move into functions of three (or more) variables it becomes impossible to graph these functions, as the graph would require four (or more) dimensions. Although given my artistic abilities I'm not sure that being able to draw something renders any conceptual advantage.

We'll focus on triple integrals today. A triple integral involves integrating a function of three variables, $f(x, y, z)$, and while as mentioned it is impossible to draw the graph of this function, it is possible to draw the domain of the function.

So, consider a function f of three variables that is defined over a box-shaped domain B. We form a partition P of the box B by passing planes through B parallel to the coordinate planes, cutting B into smaller boxes $B_1, B_2, \ldots, B_n$. Take one of these boxes, say B_i, and pick a point within it (x_i, y_i, z_i).

A picture of this idea is below:



We approximate the triple integral as:

$$\sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$$

where $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$.

If we define the norm of the partition $||P||$ to be the length of the longest diagonal of any of the boxes B_i in the partition, then we define the integral as being the limiting value of the above approximation as we take a sequence of partitions whose norm goes to 0:

$$\int\int\int_B f(x, y, z)dV = \lim_{||P||\to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k$$

provided that the limit exists. Now, again, the set of which functions are integrable, and how to prove which functions are integrable, is an interesting problem, but its mathematical difficulty is beyond this course, and furthermore proving it would take us too far afield. We will just say that functions that are continuous except for a finite number of discontinuities along smooth surfaces are integrable. This will handle all the actual examples we will encounter.

We note that the standard properties of integrals of one variable and two variables extend to integrals of three variables, and in fact to integrals of any number of variables. These properties include linearity, additivity on sets that overlap only on a boundary surface, and the comparison property.

Also, we note that triple integals can be written as triple iterated integrals. For example:

Example

Evaluate the integral:

$$\int\int\int_B x^2yz\,dV$$

$$B = \{(x,y,z)|1 \le x \le 2, 0 \le y \le 1, 0 \le z \le 2\}$$

Choosing the order $dy\,dx\,dz$

$$= \int_0^2 \int_1^2 \int_0^1 x^2 y z \, dy\, dx\, dz$$

$$= \int_0^2 \int_1^2 \left(\frac{x^2y^2z}{2} \Big|_{y=0}^{y=1} \right) dx\, dz$$

$$= \int_0^2 \int_1^2 \frac{x^2 z}{2}\, dx\, dz$$

$$= \int_0^2 \left(\frac{x^3 z}{6} \Big|_{x=1}^{x=2} \right) dz = \int_0^2 \left(\frac{8}{6}z - \frac{z}{6} \right) dz$$

$$= \int_0^2 \frac{7}{6} z\, dz = \frac{7}{12} z^2 \Big|_{z=0}^{z=2} = \boxed{\frac{7}{3}}$$

Now, if we want to integrate a function $f(x,y,z)$ over a closed bounded set S in three-space, where S is not necessarily a box, we just enclose the set S in a box B and define:

$$\tilde{f}(x,y,z)\begin{cases} f(x,y,z) & (x,y,z) \in S \\ 0 & (x,y,z) \notin S \end{cases}$$

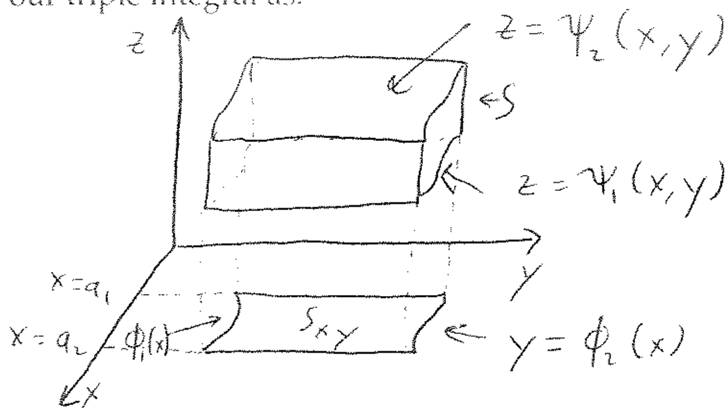
We then define the integral of $f(x,y,z)$ over S using the function $\tilde{f}$ and the box B:

$$\int\int\int_S f(x,y,z)dV = \int\int\int_B \tilde{f}(x,y,z)dV.$$

Now, if you're a good mathematician you may ask whether we can be certain this is well defined. That is, whether we can be certain that this value does not depend on the box B and the corresponding function $\tilde{f}$. The answer is yes, and in later classes you'll prove things like this. For now, don't worry about it.

Now, just as with double integrals, we can write triple integrals as iterated integrals where our limits of integration may be functions, as long as our domain D is simple with respect to one or more of our variables.

For example, if our set S is z-simple, which means that vertical lines intersect S in a single line segment if they intersect at all, then we can write our triple integral as:



$$\int\int\int_S f(x,y,z)dV = \int\int_{S_{xy}} \left[\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz\right] dA$$

where S_{xy} is the projection of the set S onto the xy-plane. If, in addition the set S_{xy} is y-simple then we can rewrite the outer double integral as an iterated integral:

$$\iiint_S f(x,y,z)\,dV = \int_{a_1}^{a_2}\int_{\phi_1(x)}^{\phi_2(x)}\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Now, other orders of integration may be possible , depending on the shape of S, but no matter what order we choose we should always get the same final value for our integral.

2 Examples

Example

Evaluate the iterated integral

$$\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx$$

$$\int_{-2}^{5}\int_{0}^{3x}\int_{y}^{x+2} 4\,dz\,dy\,dx$$

$$= \int_{-2}^{5}\int_{0}^{3x}\left(4z\,\Big|_{z=y}^{z=x+2}\right)dy\,dx$$

$$= \int_{-2}^{5}\int_{0}^{3x}\left(4(x+2) - 4y\right)dy\,dx$$

$$= \int_{-2}^{5}\left[4(x+2)y - 2y^2\,\Big|_{y=0}^{y=3x}\right]dx$$

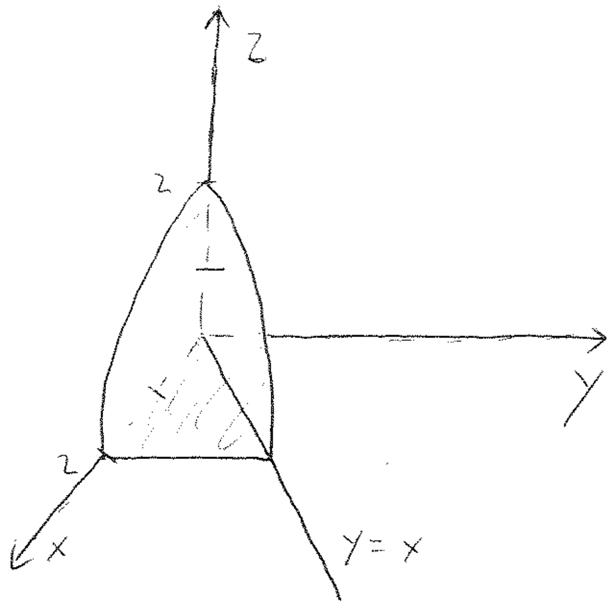
$$= \int_{-2}^{5}\left[4(x+2)(3x) - 2(3x)^2\right]dx$$

$$= \int_{-2}^{5}(-6x^2+24x)\,dx = 12x^2 - 2x^3\Big|_{x=-2}^{x=5} = \left(12(25)-2(125)\right) - \left(12(-2)^2 - 2(-2)^3\right)$$

$$= 50 - (64) = \boxed{-14}$$

Example

Evaluate the triple integral of $f(x, y, z) = 2xyz$ over the solid region S in the first octant that is bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0$, $y = x$, and $y = 0$.



We note our x-value goes from 0 to 2.

As a function of x our y-value goes from 0 to x.

As a function of x and y our z-value goes from 0 to $2 - \frac{1}{2}x^2$.

So,

$$\iiint_S 2xyz\,dV = \int_0^2 \int_0^x \int_0^{2-\frac{1}{2}x^2} 2xyz\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^x \left(xyz^2 \Big|_0^{2-\frac{1}{2}x^2} \right) dy\,dx = \int_0^2 \int_0^x xy\left(2 - \tfrac{1}{2}x^2\right)^2 dy\,dx$$

$$= \int_0^2 \left(x\left(2 - \tfrac{1}{2}x^2\right)^2 \frac{y^2}{2} \Big|_0^x \right) dx = \int_0^2 \frac{x^3}{2}\left(2 - \tfrac{1}{2}x^2\right)^2 dx$$

$$= \int_0^2 \left(2x^3 - x^5 + \tfrac{1}{8}x^7\right) dx = \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{64} \Big|_0^2 = 8 - \frac{32}{3} + 4 = \boxed{\frac{4}{3}}$$

Example

Evaluate the integral of the previous example by doing the integration in the order $dydxdz$.

The z-value goes from 0 to 2.

the x-value as a function of z goes from 0 to $\sqrt{4-2z}$.

the y-value as a function of z and x goes from 0 to x.

$$\int_0^2 \int_0^{\sqrt{4-2z}} \int_0^x 2xyz \, dy \, dx \, dz$$

$$= \int_0^2 \int_0^{\sqrt{4-2z}} \left[xy^2z \Big|_{y=0}^{y=x} \right] dx \, dz$$

$$= \int_0^2 \int_0^{\sqrt{4-2z}} x^3 z \, dx \, dz = \int_0^2 \left(\frac{x^4 z}{4} \Big|_0^{\sqrt{4-2z}} \right) dz$$

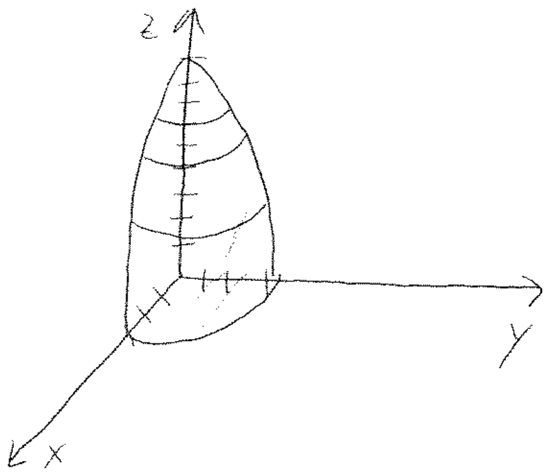
$$= \int_0^2 \frac{(4-2z)^2 z}{4} dz \qquad \cancel{u = 4-2z} \quad \cancel{du = -2d}$$

$$= \frac{1}{4} \int_0^2 (16z - 16z^2 + 4z^3) dz$$

$$= \frac{1}{4} \left(8z^2 - \frac{16}{3} z^3 + z^4 \right) \Big|_0^2 = \frac{1}{4} \left(32 - \frac{128}{3} + 16 \right) = \boxed{\frac{4}{3}}$$

Example

Calculate the volume of the region in the first octant bounded by the surface $z = 9 - x^2 - y^2$ and the coordinate planes.



$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} dz\, dy\, dx$$

$$= \int_0^3 \int_0^{\sqrt{9-x^2}} (9-x^2-y^2)\, dy\, dx$$

$$= \int_0^3 \left((9-x^2)y - \frac{y^3}{3} \Big|_0^{\sqrt{9-x^2}} \right) dx$$

$$= \int_0^3 \frac{2}{3}(9-x^2)^{3/2} = \frac{1}{12}\left(x\sqrt{9-x^2}\,(45-2x^2) + 243 \sin^{-1}\left(\frac{x}{3}\right)\right)\Big|_0^3$$

$$= \frac{243}{12}\sin^{-1}(1) = \frac{243}{12}\left(\frac{\pi}{2}\right) = \boxed{\cancel{\frac{81\pi}{16}}}$$

$$\boxed{\frac{81\pi}{8}}$$

Example
Write the iterated integral:

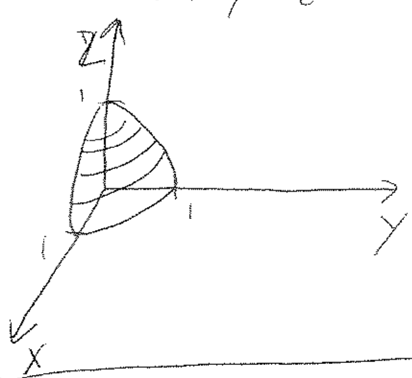
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dz\,dy$$

as an integral with the order of integration $dz\,dy\,dx$.

y goes from 0 to 1.

As a function of y z goes from 0 to $\sqrt{1-y^2}$

As a function of y and z x goes from 0 to $\sqrt{1-y^2-z^2}$

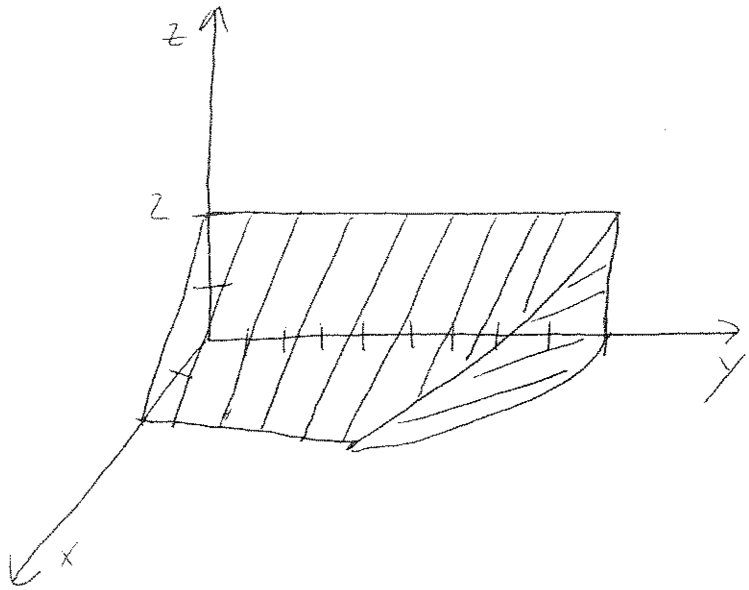


$$= \boxed{\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx}$$

Example
Write the iterated integral:

$$\int_0^2 \int_0^{9-x^2} \int_0^{2-x} f(x,y,z)\,dz\,dy\,dx$$

as an integral with the order of integratin $dydxdz$.



z goes from 0 to 2 x from 0 to 2-z

$$\boxed{\int_0^2 \int_0^{2-z} \int_0^{9-x^2} f(x,y,z)\,dy\,dx\,dz}$$

and y from 0 to $9-x^2$.