Math 2210 - Section 13.6 Surface Area

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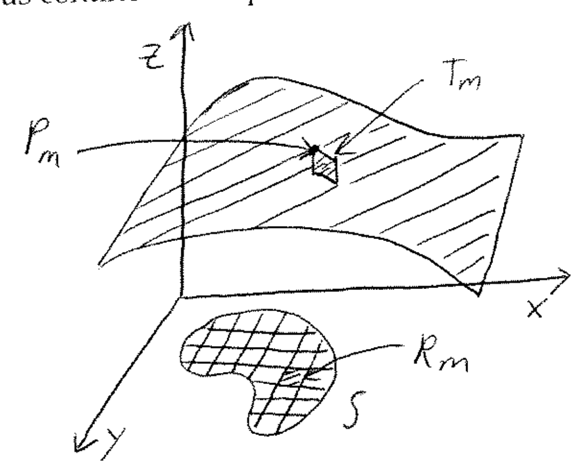
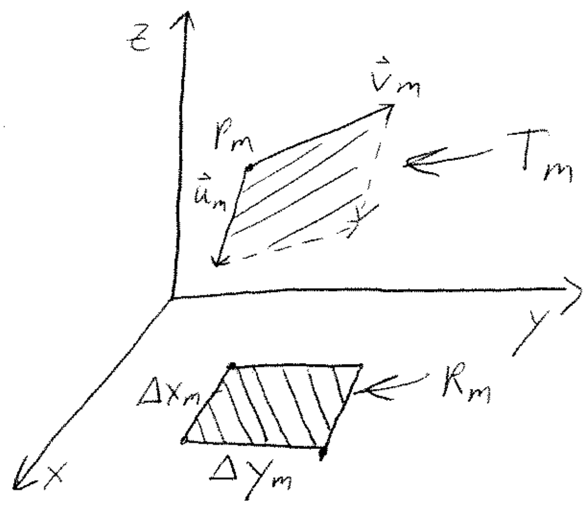
1 Derivation

Suppose we have a surface G defined over a closed and bounded region S in the xy-plane. Assume that G is defined by a function $z = f(x, y)$, and that f has continous first partial derivatives f_x and f_y.



We begin by creating a partition of the region S into lines parallel to the x and y axes, and denote by R_m the resulting rectangles that lie completely within S. For each m, let G_m be the part of the surface that projects onto R_m, and let P_m be the point of G_m that projects onto the corner of R_m with the smallest x and y coordinates. Finally, let T_m denote the parallelogram from the tangent plane at P_m that projects onto R_m.

Got all that? Basically, just take the rectangle R_m, and a point on the surface G above R_m. Find the tangent plane to the surface at that point, and take the part of this tangent plane that projects onto R_m.



The part of the tangent plane that projects onto R_m, namely T_m, is going to be a parallelogram. The sides of T_m will be formed by the vectors:

$$\begin{aligned} \mathbf{u}_m &= \Delta x_m \mathbf{i} + f_x(x_m, y_m)\Delta x_m \mathbf{k} \\ \mathbf{v}_m &= \Delta y_m \mathbf{j} + f_y(x_m, y_m)\Delta y_m \mathbf{k}. \end{aligned}$$

Now, the area of this parallelogram will be:

$$\mathbf{u}_m \times \mathbf{v}_m = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_m & 0 & f_x(x_m, y_m)\Delta x_m \\ 0 & \Delta y_m & f_y(x_m, y_m)\Delta y_m \end{vmatrix}$$

$$= \Delta x_m \Delta y_m [-f_x(x_m, y_m)\mathbf{i} - f_y(x_m, y_m)\mathbf{j} + \mathbf{k}]$$

$$= A(R_m)[-f_x(x_m, y_m)\mathbf{i} - f_y(x_m, y_m)\mathbf{j} + \mathbf{k}]$$

So, the area of T_m is therefore:

$$A(T_m) = ||\mathbf{u}_m \times \mathbf{v}_m|| = A(R_m)\sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2 + 1}.$$

To find the total area we then add up the areas of these tangent parallelograms:

$$A(G) \approx \sum_{m=1}^{N} A(T_m).$$

If we then take the limit as the norm of our partition P goes to 0 we get the total surface area:

$$\begin{aligned} A(G) &= \lim_{||P|| \to 0} \sum_{m=1}^{N} A(T_m) \\ &= \lim_{||P|| \to 0} \sum_{m=1}^{N} \sqrt{1 + [f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} A(R_m) \\ &= \int \int_S \sqrt{1 + [f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} dA \end{aligned}$$

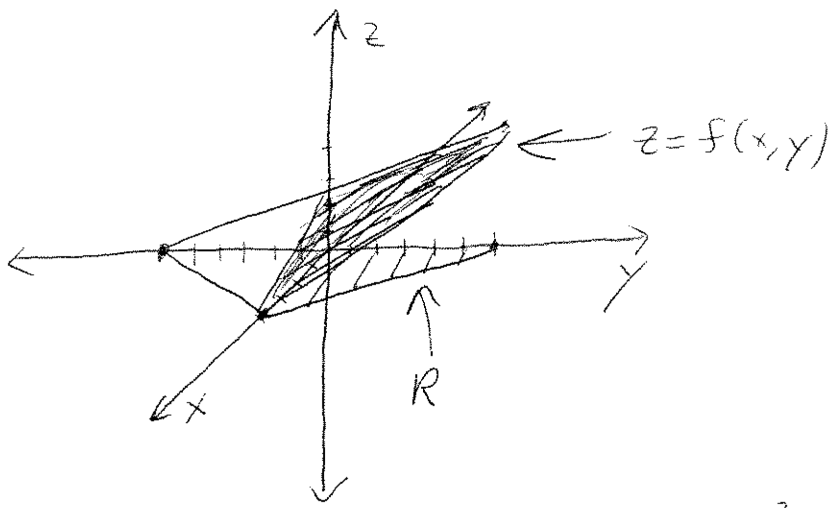
Or, more concisely,

$$A(G) = \int \int_S \sqrt{1 + f_x^2 + f_y^2} dA$$

2 Examples

Example

Find the surface area of the plane $3x - 2y + 6z = 12$ that is bounded by the planes $x = 0$, $y = 0$, and $3x + 2y = 12$.



$3x - 2y + 6z = 12$

$\Rightarrow z = 2 + \frac{1}{3}y - \frac{1}{2}x$

$f(x,y) = 2 + \frac{1}{3}y - \frac{1}{2}x$

$f_x(x,y) = -\frac{1}{2}$

$f_y(x,y) = \frac{1}{3}$

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_0^4 \int_0^{6-\frac{3}{2}x} \sqrt{1 + \left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^2}\, dy\, dx$$

$$= \int_0^4 \int_0^{6-\frac{3}{2}x} \sqrt{1 + \tfrac{1}{4} + \tfrac{1}{9}}\, dy\, dx$$

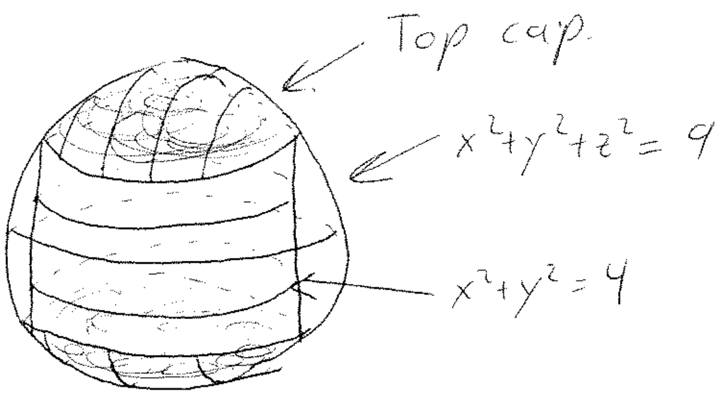
$$= \int_0^4 \int_0^{6-\frac{3}{2}x} \sqrt{\tfrac{49}{36}}\, dy\, dx = \frac{7}{6}\int_0^4 \left(6 - \tfrac{3}{2}x\right) dx$$

$$= \frac{7}{6}\left[6x - \tfrac{3}{4}x^2\right]\Big|_0^4 = \frac{7}{6}\left(6(4) - \tfrac{3}{4}(4^2)\right)$$

$$= 7(4-2) = \boxed{14}$$

Example

Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ inside the circular cylinder $x^2 + y^2 = 4$.



The surface will consist of two identical parts, both with equal area. The area of the top cap will be:

$$z = f(x,y) = \sqrt{9-x^2-y^2}$$

$$f_x(x,y) = \frac{-x}{\sqrt{9-x^2-y^2}} \qquad f_y(x,y) = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$\iint_R \sqrt{1+\left(\frac{-x}{\sqrt{9-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{9-x^2-y^2}}\right)^2}\,dA = 3\iint_R \frac{1}{\sqrt{9-x^2-y^2}}\,dA$$

~~This is best done in polar coordinates.~~

$u = 9 - r^2 \quad du = -2r\,dr$

$$= 3\int_0^{2\pi}\int_0^2 \frac{r}{\sqrt{9-r^2}}\,dr\,d\theta$$

$$= -\frac{3}{2}\int_0^{2\pi}\int_9^5 \frac{du}{\sqrt{u}}\,d\theta = \frac{3}{2}\int_0^{2\pi}\int_5^9 \frac{du}{\sqrt{u}}\,d\theta$$

$$= \frac{3}{2}\int_0^{2\pi}\left(2\sqrt{u}\right)\Big|_5^9\,d\theta = 3\int_0^{2\pi}\left(\sqrt{9}-\sqrt{5}\right)d\theta = 6\pi\left(3-\sqrt{5}\right)$$

So, the total is: $\boxed{12\pi(3-\sqrt{5})}$