# Math 2210 - Section 13.6 Surface Area

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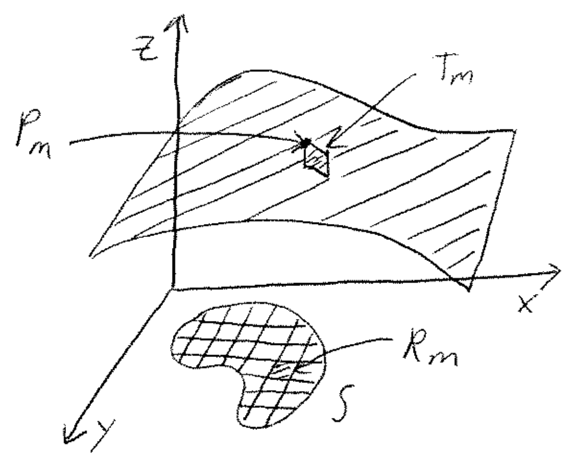
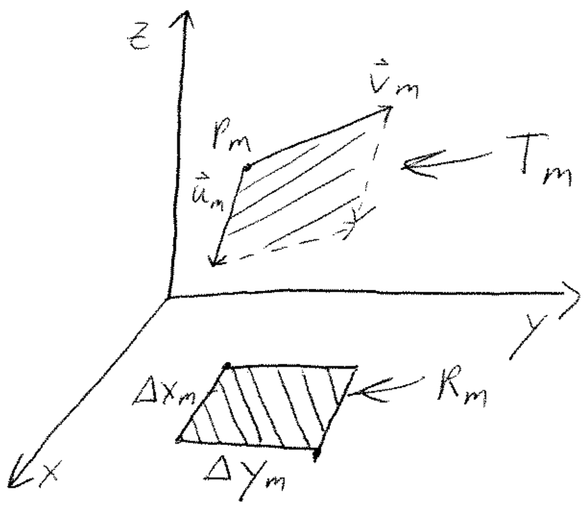
## 1 Derivation

Suppose we have a surface $G$ defined over a closed and bounded region $S$ in the $xy$-plane. Assume that $G$ is defined by a function $z = f(x, y)$, and that $f$ has continous first partial derivatives $f_x$ and $f_y$.




We begin by creating a partition of the region $S$ into lines parallel to the $x$ and $y$ axes, and denote by $R_m$ the resulting rectangles that lie completely within $S$. For each $m$, let $G_m$ be the part of the surface that projects onto $R_m$, and let $P_m$ be the point of $G_m$ that projects onto the corner of $R_m$ with the smallest $x$ and $y$ coordinates. Finally, let $T_m$ denote the parallelogram from the tangent plane at $P_m$ that projects onto $R_m$.

Got all that? Basically, just take the rectangle $R_m$, and a point on the surface $G$ above $R_m$. Find the tangent plane to the surface at that point, and take the part of this tangent plane that projects onto $R_m$.




The part of the tangent plane that projects onto $R_m$, namely $T_m$, is going to be a parallelogram. The sides of $T_m$ will be formed by the vectors:

$$\mathbf{u}_m = \Delta x_m \mathbf{i} + f_x(x_m, y_m)\Delta x_m \mathbf{k}$$
$$\mathbf{v}_m = \Delta y_m \mathbf{j} + f_y(x_m, y_m)\Delta y_m \mathbf{k}.$$

Now, the area of this parallelogram will be:

$$\mathbf{u}_m \times \mathbf{v}_m = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x_m & 0 & f_x(x_m, y_m)\Delta x_m \\ 0 & \Delta y_m & f_y(x_m, y_m)\Delta y_m \end{vmatrix}$$

$$= \Delta x_m \Delta y_m [-f_x(x_m, y_m)\mathbf{i} - f_y(x_m, y_m)\mathbf{j} + \mathbf{k}]$$

$$= A(R_m)[-f_x(x_m, y_m)\mathbf{i} - f_y(x_m, y_m)\mathbf{j} + \mathbf{k}]$$

So, the area of $T_m$ is therefore:

$$A(T_m) = ||\mathbf{u}_m \times \mathbf{v}_m|| = A(R_m)\sqrt{[f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2 + 1}.$$

To find the total area we then add up the areas of these tangent parallelograms:

$$A(G) \approx \sum_{m=1}^{N} A(T_m).$$

If we then take the limit as the norm of our partition $P$ goes to 0 we get the total surface area:

$$A(G) = \lim_{||P|| \to 0} \sum_{m=1}^{N} A(T_m)$$

$$= \lim_{||P|| \to 0} \sum_{m=1}^{N} \sqrt{1 + [f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} A(R_m)$$

$$= \int \int_S \sqrt{1 + [f_x(x_m, y_m)]^2 + [f_y(x_m, y_m)]^2} dA$$

Or, more concisely,

$$A(G) = \int \int_S \sqrt{1 + f_x^2 + f_y^2} dA$$

## 2 Examples

*Example*

Find the surface area of the plane $3x - 2y + 6z = 12$ that is bounded by the planes $x = 0$, $y = 0$, and $3x + 2y = 12$.

*Example*

Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 9$ inside the circular cylinder $x^2 + y^2 = 4$.