Math 2210 - Section 13.4 Double Integrals in Polar Coordinates

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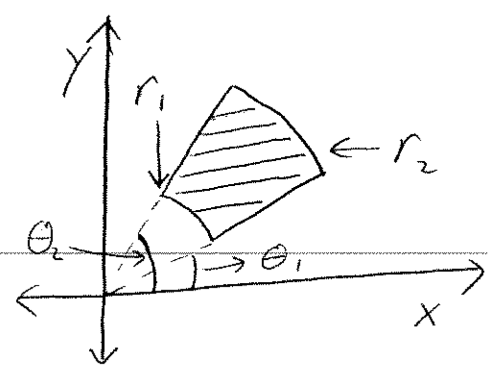
1 Double Integrals in Polar Coordinates

Some curves in the plane can most easily be defined using polar coordinates. These curves include circles, cardioids, and roses. So, as you might expect, double integrals over regions defined by these curves are more easily evaluated if we switch to polar coordinates. However, the question that naturally arises is, how do we perform double integrals using polar coordinates? Well, today we'll talk about how to do this.

1.1 Polar Rectangles

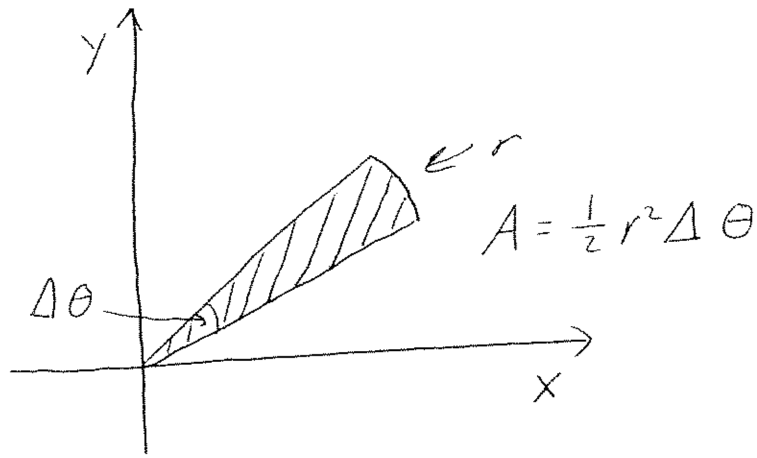
First off, instead of looking at rectangular regions, we need to look at regions called "polar rectangles". These are regions that look like this:



$$R = \{(r, \theta) | r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}.$$

Note that we require $r_1 \geq 0$ and $\theta_2 - \theta_1 \leq 2\pi$.

Now, the area of a "slice of pizza" like this:



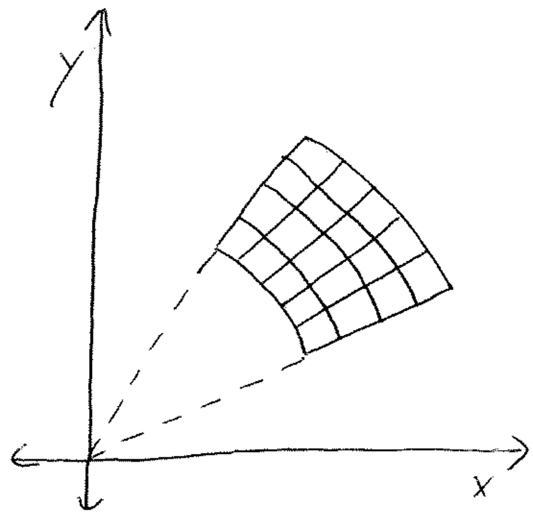
is $\frac{1}{2}r^2\Delta\theta$. So, a polar rectangle is the difference between the areas of two slices of pizza, and therefore has the area:

$$\frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}(r_2 + r_1)\Delta r\Delta\theta = \bar{r}\Delta r\Delta\theta.$$

where $\bar{r}$ is the average value of r_1 and r_2.

1.2 The Double Integral in Polar Coordinates

So, how do we take a double integral in polar coordinates? Well, if we're integrating over a polar rectangle, we just partition the polar rectangle up into smaller rectangles:



and then approximate the volume as:

$$V \approx \sum_{k=1}^{n} f(\overline{r_k}, \overline{\theta_k})\overline{r_k}\Delta r_k \Delta\theta_k$$

Then, we do what we always do. We take the limit as these partitions become finer and finer, and the limit is defined as our total volume, or in other words, our double integral.

$$V = \int \int_R f(r, \theta) r dr d\theta$$

where we note that if our function f is defined in terms of the variables x and y, so $f(x, y)$, we can rewrite it as a function of r and θ using the relations $x = r\cos\theta$ and $y = r\sin\theta$, creating $f(r\cos\theta, r\sin\theta)$.

Now, we've derived this result under the assumtion that f is nonnegative, but it's also valid in general, as long as our function f is integrable, and we interpret the volume of regions under the xy-plane as "negative volume".

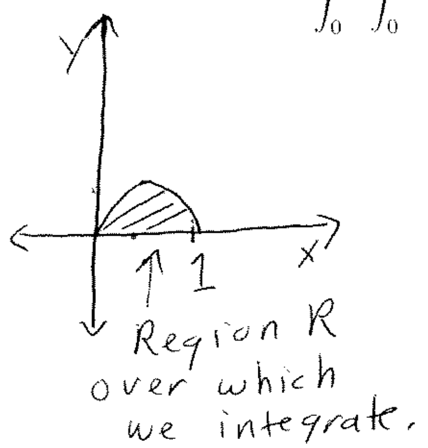
1.3 Iterated Integrals

The result above becomes useful when we write our polar integral as an iterated integral. Here's an example.

Example

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r^2 \sin\theta dr d\theta$$

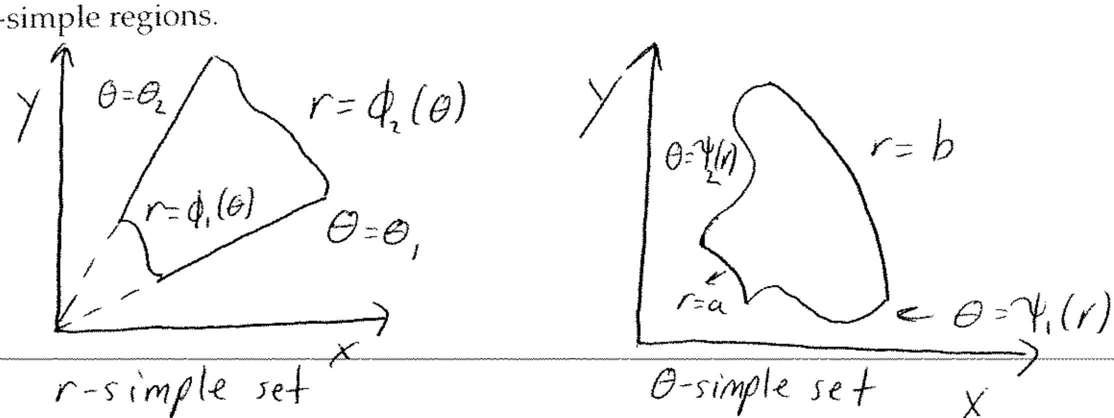


$$\int_0^{\pi/2}\int_0^{\cos\theta} r^2\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\sin\theta\Big|_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\cos^3\theta\sin\theta}{3}\, d\theta = -\frac{\cos^4(\theta)}{12}\Big|_0^{\pi/2} = \boxed{\frac{1}{12}}$$

Now, so far we've only dealt with integrals over polar rectangles, which are a pretty restrictive class of domains. However, we can extend our definition to deal with r-simple and θ-simple regions like the ones depicted below in pretty much exactly the same way as we did for x-simple and y-simple regions.



I'll omit the details and just skip to a few examples:

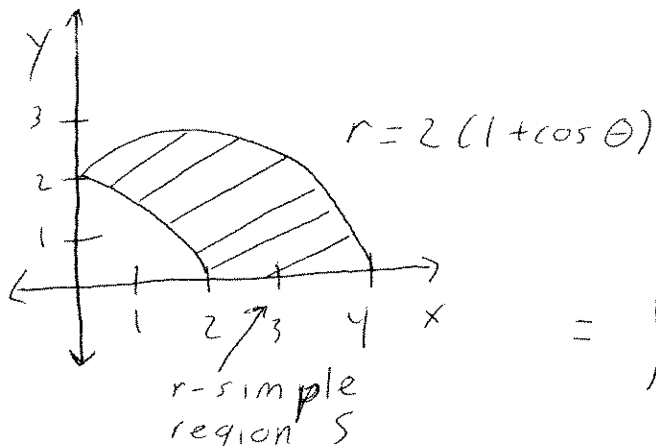
1.4 Examples

Example

Evaluate the integral:

$$\int\int_S y\,dA$$

where S is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.



$$= \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3} \sin\theta \Big|_2^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\left(\left(1+\cos\theta\right)^3 - 1\right) \sin\theta \, d\theta$$

$u = 1 + \cos\theta$

$du = -\sin\theta$

$$= \frac{8}{3}\int_2^1 (-1)\left[u^3 - 1\right] du$$

$$= \frac{8}{3}\int_1^2 (u^3 - 1)\,du = \frac{8}{3}\left[\frac{u^4}{4} - u\right]\Big|_1^2$$

$$= \frac{8}{3}(4-2) - \frac{8}{3}\left(\frac{1}{4} - 1\right)$$

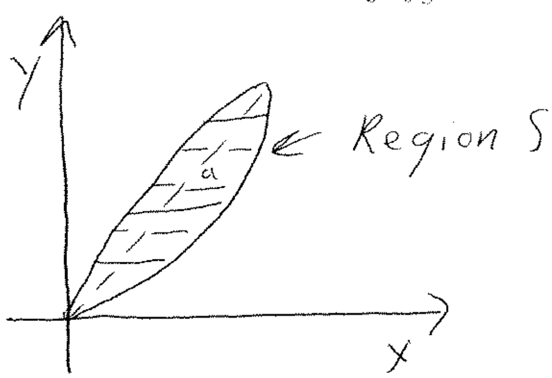
$$= \frac{8}{3}\left[2 + \frac{3}{4}\right] = \boxed{\frac{22}{3}}$$

Example
Evaluate the integral:

$$\int\int_S r dr d\theta$$

where S is one leaf of the four-leaved rose $r = a\sin 2\theta$.

Note: the integral $\int\int_S r dr d\theta$ calculates the area of the region S.



$$\int_0^{\pi/2}\int_0^{a\sin 2\theta} r dr d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{a\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{a^2 \sin^2 2\theta}{2} d\theta \qquad \sin^2(2\theta) = \frac{1-\cos 4\theta}{2}$$

$$= \frac{a^2}{4}\int_0^{\pi/2}(1-\cos 4\theta) d\theta = \frac{a^2}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]\Big|_0^{\pi/2}$$

$$= \frac{a^2}{4}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi a^2}{8}}$$

1.5 The Area Under the Normal Distribution Curve

If you try to take the antiderivative of the function $f(x) = e^{-x^2}$ you will fail. More precisely, you cannot express the antiderivative of $f(x)$ in terms of the standard functions we all know and love like polynomials, rational functions, logarithms, exponentials, and triginometric functions. In fact, the antiderivative is a new function called the "error function", Erf, and pronounced "urf". So, not knowing anything about the properties of the error function, suppose I asked you to calculate:

$$\int_{-\infty}^{\infty} e^{-x^2}dx$$

how would you do it? Well, you probably wouldn't, but using polar integrals you can calculate this using one of the most ingenious little tricks in all of mathematics. This one's a keeper. Here's what you do. You note that:

$$\left[\int_{-\infty}^{\infty} e^{-x^2}dx\right]^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$$

How the heck is this any easier? We've taken a single integral and made it into a double integral. Doesn't that make it harder? Well, check this out. Basically, what we're doing is integrating over all of $\mathbb{R}^2$, and in polar coordinates that would be when r goes from 0 to ∞ and θ goes from 0 to 2π. So, we can transform the above integral into an integral in polar coordinates:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy = \int_{0}^{2\pi}\int_{0}^{\infty} re^{-r^2}drd\theta.$$

Now are you seeing it? That factor of integration r is here to save the day. From this point on the integral is easy. We just do a u substitution of $u = r^2$ to get:

$$\int_0^{2\pi}\int_0^{\infty} re^{-r^2}drd\theta = \int_0^{2\pi}\int_0^{\infty}\frac{1}{2}e^{-u}dud\theta.$$

Now, we can integrate this no problem to get:

$$\int_0^{2\pi}\int_0^{\infty}\frac{1}{2}e^{-u}dud\theta = \int_0^{2\pi} -\frac{1}{2}e^{-u}|_0^{\infty}d\theta = \int_0^{2\pi}\frac{1}{2}d\theta = \pi.$$

So, if we walk back the cat on our series of equalities we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}dxdy = \sqrt{\pi}.$$

Amazing! Also, if you've taken a statistics class, you may have seen the normalized normal distribution curve written as:

$$N(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

and you may have wondered, if you wonder about these things (and you should) why the heck there was a $\sqrt{2\pi}$ term in there. Well, now you know. As an exercise, you should try to confirm that:

$$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = 1.$$
