Math 2210 - Section 13.3 Iterated Integrals

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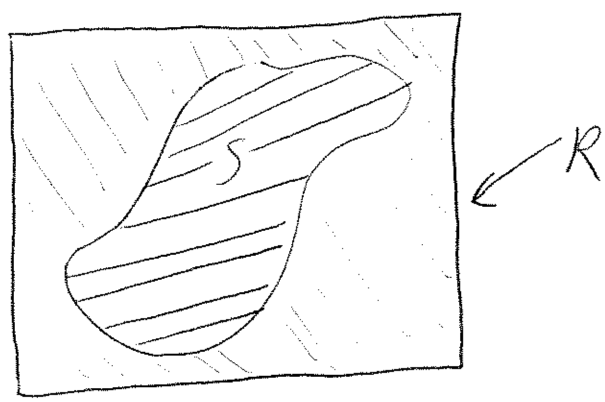
1 Double Integrals over Nonrectangular Regions

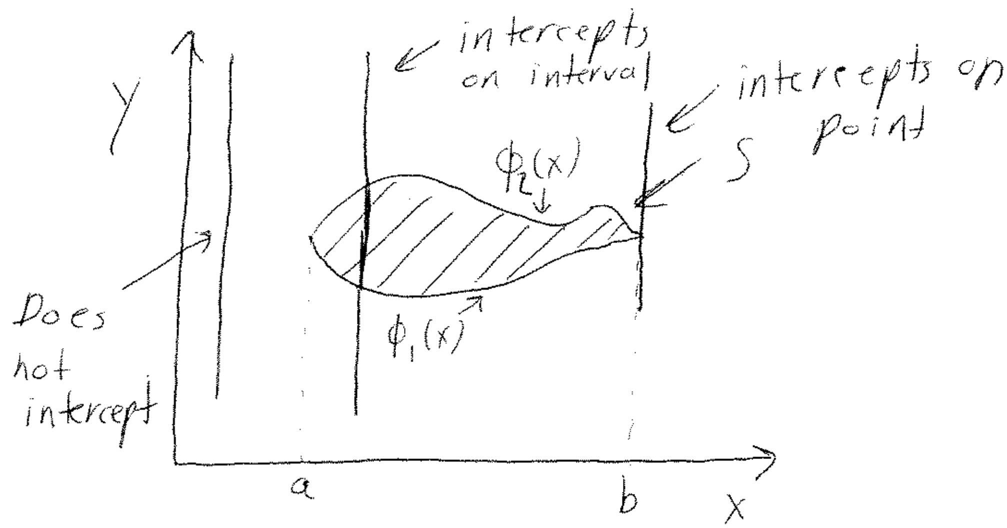
1.1 Double Integrals over y-Simple or x-Simple Regions

Consider an arbitrary closed and bounded set S in the plane. If we surround the set S with a rectangle R with sides parallel to the coordinate axes, then we can define (or redefine, if necessary) $f(x, y) = 0$ on the part of R outside of S. We say that f is integrable on S if it is integrable on R, and we write:



$$\int\int_S f(x, y)dA = \int\int_R f(x, y)dA.$$

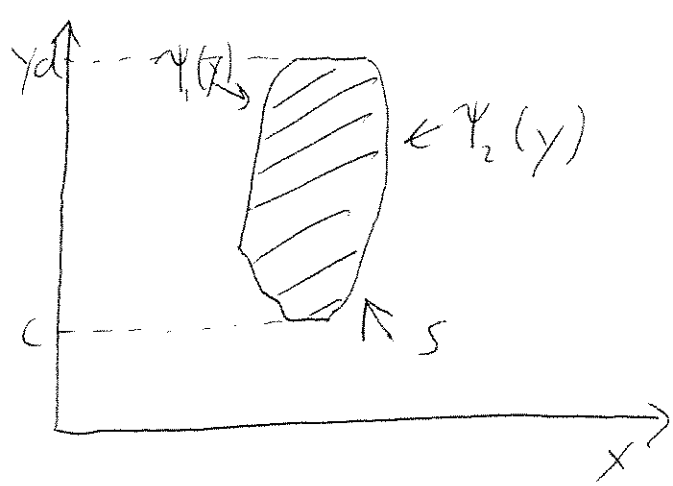
Now, in general, an arbitrary closed and bounded set S can be very complicated. So, we're goint to restrict ourselves to just a subset of these possible sets. Namely, we're going to restrict ourselves to either x-simple or y-simple sets. A set S is y-simple if any line parallel to the y-axis intersects S either in a single interval, at a point, or not at all.



So, a set S is y-simple if there are functiosn $\phi_1(x)$ and $\phi_2(x)$ on $[a, b]$ such that:

$$S = \{(x, y) : \phi_1(x) \leq y \leq \phi_2(x), a \leq x \leq b\}.$$

A set S being x-simple is defined analogously.



Now, if we want to evaluate the double integral of a function $f(x, y)$ over a y-simple set S, we first enclose S in a rectangle R and make $f(x, y) = 0$ outside of S. Then using our earlier definition we get:

$$\int\int_S f(x, y)dA = \int\int_R f(x, y)dA = \int_a^b \left[\int_c^d f(x, y)dy\right] dx = \int_a^b \left[\int_{o_1(x)}^{o_2(x)} f(x, y)dy\right] dx.$$

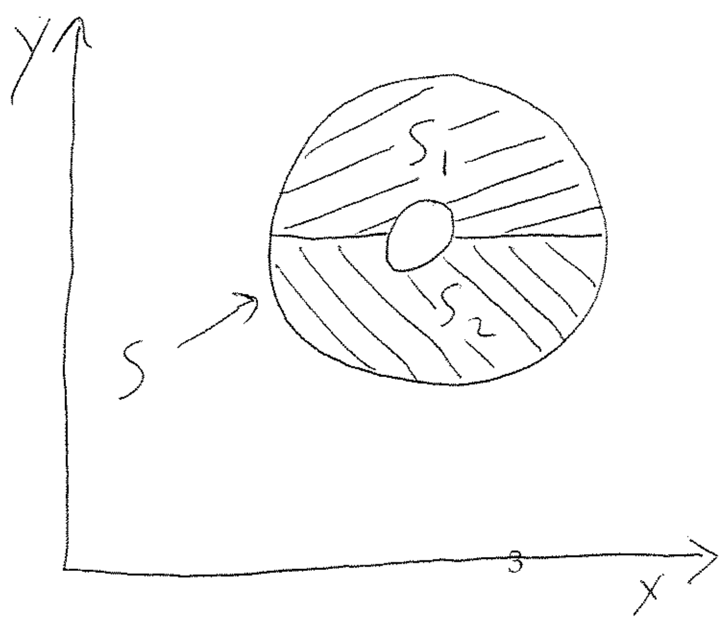
which, eliminating the middle terms gives us:

$$\int\int_S f(x, y)dA = \int_a^b \int_{o_1(x)}^{o_2(x)} f(x, y)dydx.$$

If the set S is x-simple, identical reasoning gives us the formula:

$$\int\int_S f(x, y)dA = \int_c^d \int_{\upsilon_1(y)}^{\upsilon_2(y)} f(x, y)dxdy.$$

Now, if S is neither x-simple nor y-simple, we can frequently divide it up into regions that are, and then integrate these regions individually, and then add up all the individual integrals to get the total integral.



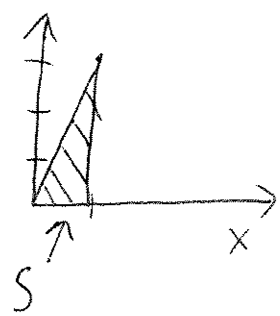
Divide S into S_1 and S_2, both y-simple

1.2 Examples

Example

Evaluate the integral:

$$\int_0^1 \int_0^{3x} x^2 dydx$$



$$\int_0^1 \int_0^{3x} x^2 dy\, dx$$

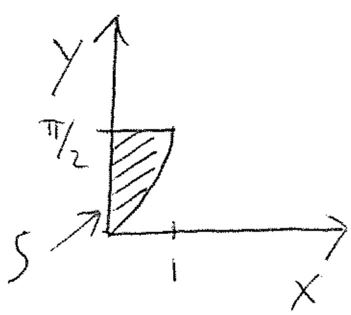
$$= \int_0^1 \left[yx^2 \Big|_0^{3x} \right] dx$$

$$= \int_0^1 3x^3 dx = \frac{3}{4}x^4 \Big|_0^1 = \boxed{\frac{3}{4}}$$

Example

Evaluate the integral:

$$\int_0^{\frac{\pi}{2}} \int_0^{\sin y} e^x \cos y dxdy$$



$$\int_0^{\pi/2} \int_0^{\sin y} e^x \cos y \, dx dy$$

$$= \int_0^{\pi/2} \left(e^x \cos y \Big|_0^{\sin y} \right) dy$$

$$= \int_0^{\pi/2} \left(e^{\sin y} \cos y - \cos y \right) dy$$

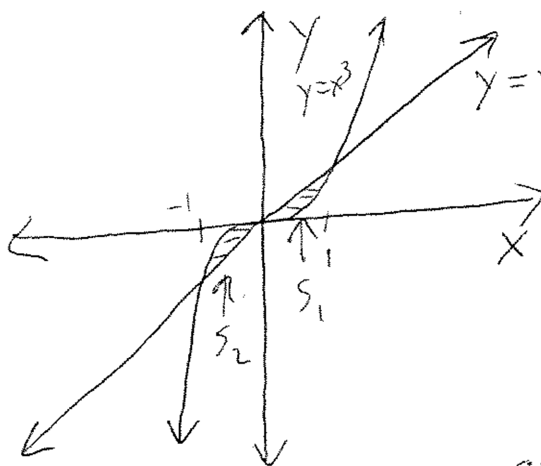
$$= e^{\sin y} - \sin y \Big|_0^{\pi/2} = (e-1) - (1-0)$$

$$= \boxed{e-2}$$

Example
Evaluate the integral:

$$\int\int_S x\,dA$$

where S is the region between $y = x$ and $y = x^3$.



$$= \int_{-1}^{0}\int_{x}^{x^3} x\,dy\,dx + \int_0^1\int_{x^3}^{x} x\,dy\,dx$$

$S = S_1 \cup S_2$

$$= \int_{-1}^{0}\left(xy\Big|_{x}^{x^3}\right)dx + \int_0^1\left(xy\Big|_{x^3}^{x}\right)dx$$

$$= \int_{-1}^{0}(x^4 - x^2)\,dx + \int_0^1 (x^2 - x^4)\,dx$$

$$= \frac{x^5}{5} - \frac{x^3}{3}\Big|_{-1}^{0} + \frac{x^3}{3} - \frac{x^5}{5}\Big|_0^1$$

$$= (0-0) - \left(-\frac{1}{5} + \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - (0-0)$$

$$= \boxed{0}$$

Note: the integral over just S_2 is $2/15$, the integral over just S_1 is $-2/15$.