Math 2210 - Section 13.2 Iterated Integrals

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1 Iterated Integrals

1.1 The Idea Behind Iterated Integrals

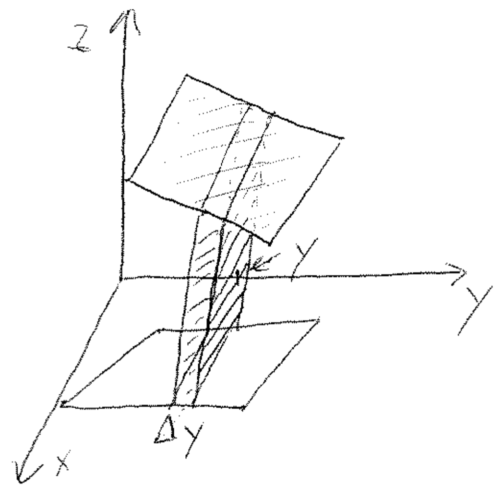
In the last section we learned how the double integral over a rectangular region is defined in terms of Riemann sums, and how these sums can be used to approximate double integrals. Now, humanity probably wouldn't have progressed as far as it has if taking a formal limit of Riemann sums was the only way to evaluate integrals. (Seriously.) The nice thing is that, just like in single variable calculus, we have ways of evaluating double integrals that are much, much easier than taking a formal Riemann sum. In fact, they're basically just a modification of the same methods we used in single variable calculus. Now, I must warn you as we start chapter 13 that all those methods of integration covered in calculus II will be important in this chapter, and so if you don't know them you should brush up.

Now, suppose for the sake of simplicity here at the start that our function $z = f(x, y)$ is strictly positive over the rectangle R, so we can intepret the double integral:

$$\int\int_R f(x, y)dA$$

over the region R as the volume under the surface $z = f(x, y)$ and above the rectangular region R.

Here is a picture illustrating the idea:



Now, if we move out a distance y along the y-axis we can slice the surface around the point y with a slice of width Δy. The volume of this slice will be approximately:

$$V(y) \approx A(y)\Delta y$$

where $A(y)$ is the area of the cross section of the solid with the plane at y parallel to the xz coordinate plane. See the above picture.

Now, if we take the limit at Δy goes to 0, and consequently the number of slices goes to infinity, we get the integral:

$$V = \int_c^d A(y)dy$$

Now, the area function $A(y)$ can be calculated by means of a single integral:

$$A(y) = \int_a^b f(x, y)dx$$

where in this integral we integrate with respect to x and just treat y as a constant. Essentially the inverse of taking a partial derivative with respect to x.

If we combine these two observations, we get:

$$V = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.$$

This is called an iterated integral.

If we recognize this volume as being the double integral over R, we get the relation:

$$\int \int_R f(x, y) dA = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.$$

Finally, we note there was nothing special about choosing to slice along the y-axis and not the x-axis, and so we get that the order of integration doesn't matter:

$$\int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$$

Formally proving this equivalence is actually pretty difficult, and in measure theory it's a result known as Fubini's thoerem. However, I hope you'll agree it's intuitively obvious.

Before we work a couple of examples, we should note that while we assumed here that $f(x, y)$ is nonnegative the formulas above are valid in general. We just need to understand that volume under the xy-plane is considered "negative volume" in the way that area below the x-axis was considered "negative area" with single integrals.

1.2 Examples

Example

Evaluate the integrals

$$\int_1^2 \int_0^3 (xy + y^2)dxdy$$

Example

$$\int_0^{\ln 3} \int_0^1 xye^{xy^2} dydx$$

Example
Evaluate:

$$\int_0^{\sqrt{3}} \int_0^1 \frac{8x}{(x^2 + y^2 + 1)^2} dy dx$$

Hint - Reverse the order of integration.