Math 2210 - Section 13.1 Notes

Dylan Zwick

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1 Double Integrals over Rectangles

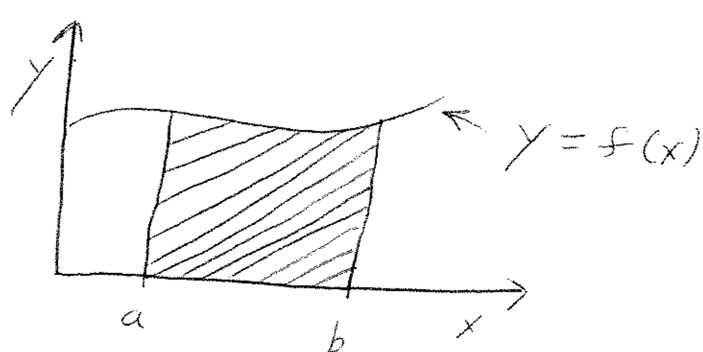
1.1 Riemann Sums

The two major tools that we learn how to use in calculus are differentiation and integration. So far in calculus III we've dealt exclusively with differentiation and problems involved with differential calculus like maxima and tangent planes. In this lecture, we begin our discussion of integration in multivariable calculus.

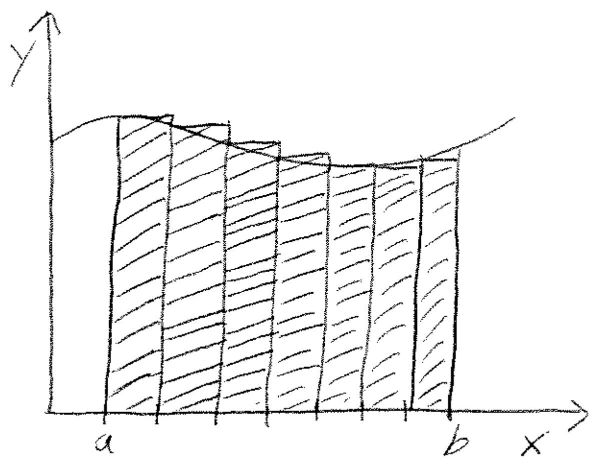
If we remember back to single variable calculus, we remember that for a function (we'll say for now a non-negative function) $f(x)$ the integral of the function from a to b:

$$\int_a^b f(x)dx$$

represented the area under the curve $f(x)$ from a to b.



Now, if we think back to how we defined this area, it began with the definition of the integral in terms of Riemann sums. The idea was that you break up the interval from a to b into a bunch of smaller segments, and then you draw a bunch of rectangles whose bases are the segments, and whose heights are the valuess of the function $f(x)$ over points within the segments. The method of choosing the points within the segments (it could be left sides, right sides, midpoints, or any other method) doesn't really matter.



If we then add up the area of these rectangles we get an estimate of the total area under the curve. Any way of chopping up the segment from a to b is called a partition, P, and the norm of the partition $||P||$ is defined as the length of the longest segment within the partition. Formally, the integral was defined as the limit of the area approximations for any sequence of partitions whose norm went to 0:

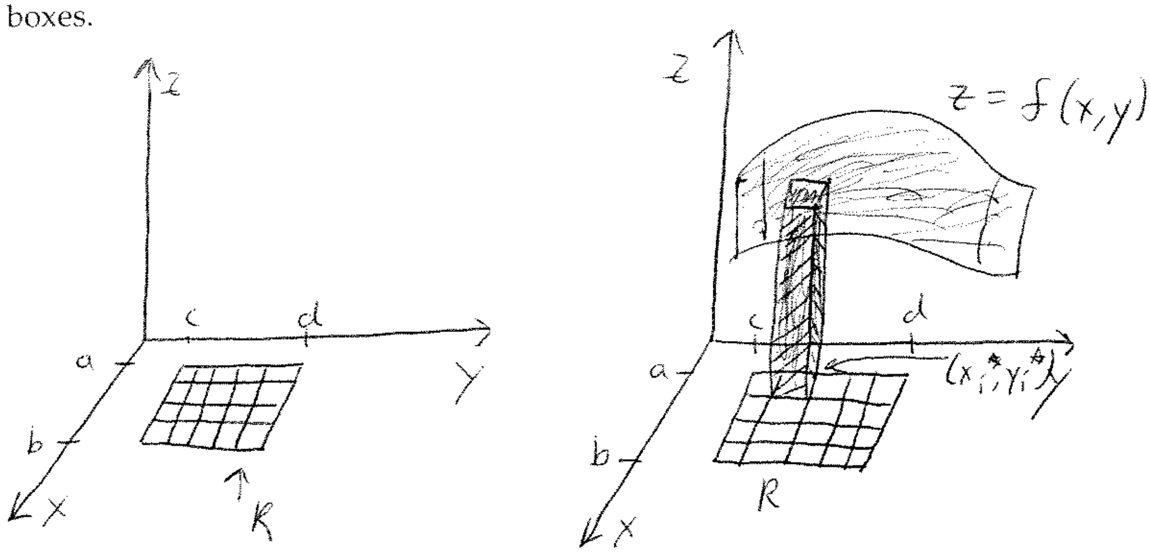
$$\int_a^b f(x)dx = \lim_{||P|| \to 0} \sum_{i=1}^{N} f(x_i^*)\Delta x_i$$

where N is the number of segments, Δx_i is the length of segment i in the partition, and x_i^* is any point within segment i. Now, the integral was well defined if this limit was well defined for any sequence of partitions whose norm went to 0, and this turns out to be the case for many functions.

In fact, it's always true for functions that are piecewise continuous from a to b.

1.2 Riemann Sums for Double Integrals

Now, in multivariable calculus the function $f(x, y)$ defines a surface $z = f(x, y)$, and the problem we will deal with first is how to calculate the volume of the region underneath the function $f(x, y)$ but above a given rectangular domain D. Just like we did with integrals of single variable functions, we will be adding up the volumes of a bunch of rectangular boxes.



We chop up our domain R into a bunch of smaller rectangles, and these rectangles form the bases for our rectangular boxes. The heights of these rectangular boxes are given by $f(x_i^*, y_i^*)$, where the points (x_i^*, y_i^*) are points in the respective rectangles R_i. We add up the volumes of all the boxes to get an estimate of the volume under the surface $f(x, y)$ and over the region R.

1.3 Definitions and Properties

The formal definition of the double integral is similar to that for the single integral:

Definition - Let $z = f(x, y)$ be defined over a closed rectangle R. If we chop up R into a bunch of smaller rectangles, this defines a partition P, and we define the norm of the partition P, denoted $||P||$, to be the maximum value of the diagonal of any of the rectangles in the partition. If the limit

$$\lim_{||P|| \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)A_i,$$

where N is the number of subrectangles, (x_i^*, y_i^*) is a point inside subrectangle R_i, and A_i is the area of subrectangle R_i, exists then $f(x, y)$ is integrable over R and we define the double integral as:

$$\int\int_R f(x, y)dA = \lim_{||P|| \to 0} \sum_{i=1}^{N} f(x_i^*, y_i^*)A_i$$

Now, just as we said that any function that is piecewise continuous on an interval is integrable on that interval, we have the following integrability theorem for multiple integrals.

Theorem - If f is bounded on the closed rectangle R and if it is continuous there, except for on a finite number of smooth curves, then f is integrable on R. If f is continuous on all of R, then f is integrable there.

Properties of the Double Integral

1. *Linearity*

   (a) $$\int\int_R kf(x, y)dA = k\int\int_R f(x, y)dA$$

   (b) $$\int\int_R [f(x, y) + g(x, y)]dA = \int\int_R f(x, y)dA + \int\int_R g(x, y)dA.$$

2. *Additivity on Rectangles*

$$\int\int_R f(x, y)dA = \int\int_{R_1} f(x, y)dA + \int\int_{R_2} f(x, y)dA$$

where R_1 and R_2 combine to form R and R_1 and R_2 overlap only perhaps on a line segment.

3. *Comparison Property*

   If $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, then

$$\int\int_R f(x, y)dA \leq \int\int_R g(x, y)dA$$

1.4 Examples

Example

For

$$f(x,y) = \begin{cases} -1 & 1 \le x \le 4, 0 \le y \le 1 \\ 2 & 1 \le x \le 4, 1 < y \le 2 \end{cases}$$

find $\int\int_R f(x,y)dA$ where $R = \{(x,y)|1 \le x \le 4, 0 \le y \le 2\}$.

$$V = -1(4-1)(1-0) + 2(4-1)(2-1)$$

$$= -3 + 6 = \boxed{3}$$

As the height is constant over the rectangles

$R_1 = \{(x,y) \mid 1 \le x \le 4, 0 \le y \le 1\}$

$R_2 = \{(x,y) \mid 1 \le x \le 4, 1 < y \le 2\}$

the volume will just be the sum of the heights over the rectangles, multiplied by the rectangles' respective areas.

Example

Let $R = \{(x, y)|0 \le x \le 6, 0 \le y \le 4\}$ and $f(x, y) = 10 - y^2$. Partition R into 6 equal squares by lines $x = 2$, $x = 4$, and $y = 2$. Approximate $\int\int_R f(x, y)dA$ as $\sum_{k=1}^{6} f(\overline{x}_k, \overline{y}_k)\Delta A_k$ where $(\overline{x}_k, \overline{y}_k)$ are the centers of the squares.

$f(1,1) = 9 \quad f(3,1) = 9 \quad f(5,1) = 9$

$f(1,3) = 1 \quad f(3,3) = 1 \quad f(5,3) = 1$

$\Delta A = 2 \times 2 = 4$ for each square

$$\iint_R f(x,y)\,dA = 4(9+9+9+1+1+1) = \boxed{120}$$

The actual value is:

$$\int_0^4\int_0^6 (10-y^2)\,dx\,dy = \int_0^4 6(10-y^2)\,dy$$

$$= 60y - 2y^3\Big|_0^4 = 240 - 128 = \boxed{112}$$