Math 2210 - Section 12.9 Notes

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1 Lagrange Multipliers

1.1 Optimization with Constraints

Quite frequently in real world problems, especially in economic problems, you see situations where you want to optimize some function subject to constraints. For example, you may want to optimize profit, subject to constraints of space and labor. Another example is given below.

Example
Find the minimum distance from the surface $z^2 = x^2y + 4$ to the origin.

$d^2(x, y, z) = x^2 + y^2 + z^2$

So, if $z^2 = x^2y + 4$ the distance as a function of x and y is:

$d^2(x, y) = x^2 + y^2 + x^2y + 4$

$d^2_x(x, y) = 2x + 2xy \quad d^2_y(x, y) = 2y + x^2$

$d^2_x(x, y) = 0$ if $x = 0$ or $y = -1$ $\quad d^2_y(x, y) = 0$ if $x = 0$ then $y = 0$, if $y = -1$ then $x = \sqrt{2}$.

$d^2_{xx}(x, y) = 2 + 2y \quad D(0, 0) = 4 \quad d^2_{xx}(x, y) > 0$

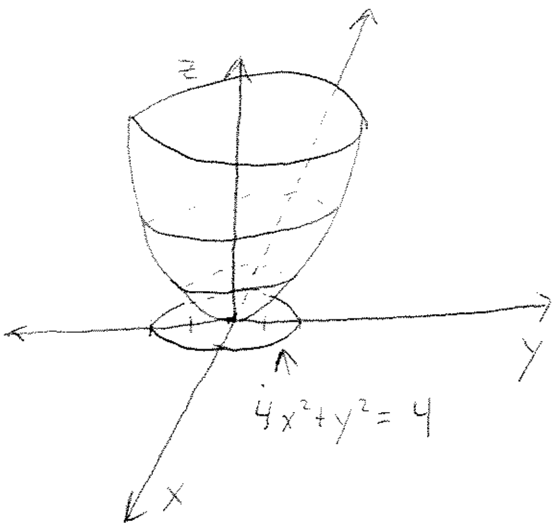
$d^2_{yy}(x, y) = 2 \quad D(\sqrt{2}, -1) = -8$

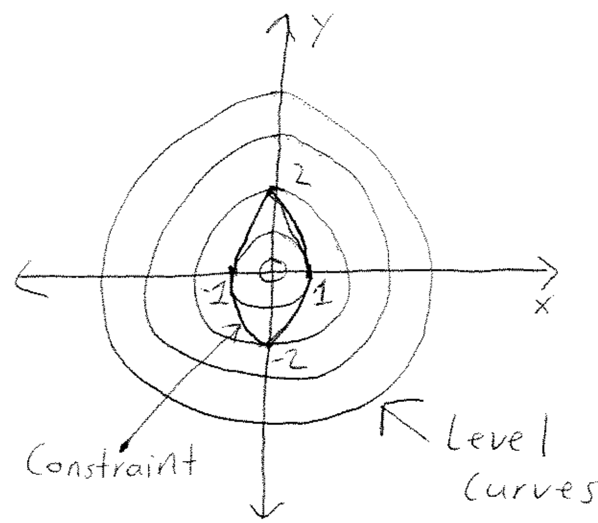
$d^2_{xy}(x, y) = 2x$

So, the minimum distance is at $(0, 0, 2)$ with value 2.

In the previous example we were able to solve the problem using substitution, but this isn't always possible, and even when it is possible it can be much easier to solve the problem using a technique known as Lagrange multipliers.

1.2 The Method of Lagrange Multipliers





Suppose that we wanted to maximize the value of the function $f(x, y) = x^2 + y^2$ subject to the constraint that x and y satisfy the relation $4x^2 + y^2 = 4$. As we can see from the graph above, if we view the level curves of the function $f(x, y)$ on the (x, y) plane we're going to want to move along our constraint curve until we reach a point where any more movement along the constraint, in any direction, would just decrease the value of the function. This will occur when the tangent line of the constraint curve is parallel to the tangent line of the level curve.

Now, this description is admittedly qualitative, and a formal proof of the method of Lagrange multipliers is pretty difficult stuff. In fact, it's even skipped in Tom Apostol's calclulus textbook, which is about as advanced as calculus textbooks come. However, while the proof of the method may be difficult, understanding the reasoning behind the proof isn't too hard, nor is using the method itself.

Now, if we express the constraint as being of the form $g(x, y) = 0$, then at a point (x_0, y_0) the tangent line to the level curve of the function $f(x, y)$ at that point will be parallel to the tangent line of the constraint if the gradient vectors $\bigtriangledown f(x, y)$ and $\bigtriangledown g(x, y)$ are parallel. Stated mathematically, this is Lagrange's method.

Theorem (Lagrange's Method) -

To maximize or minimize $f(\mathbf{p})$ subject to the constraint $g(\mathbf{p}) = 0$, solve the system of equations

$$\bigtriangledown f(\mathbf{p}) = \lambda \bigtriangledown g(\mathbf{p}) \text{ and } g(\mathbf{p}) = 0$$

for $\mathbf{p}$ and λ. Each such point $\mathbf{p}$ is a critical point for the constrained extremum problem, and the corresponding λ is called a Lagrange multiplier.

Example

Find the maximum of the function

$$f(x, y) = xy$$

subject to the constraint

$$g(x, y) = 4x^2 + 9y^2 - 36 = 0.$$

$\nabla f(x,y) = \langle y, x \rangle$

$\nabla g(x,y) = \langle 8x, 18y \rangle$

$y = \lambda 8x$, $x = \lambda 18y$ $\Rightarrow y = 144\lambda^2 y \Rightarrow \lambda = \frac{1}{12}$

$\Rightarrow y = \frac{2}{3}x \Rightarrow 4x^2 + 9\left(\frac{2}{3}x\right)^2 = 36$

$\Rightarrow 8x^2 = 36 \Rightarrow x = \pm\frac{6}{2\sqrt{2}} = \pm\frac{3}{\sqrt{2}}$

$18 + 9y^2 = 36 \Rightarrow y = \pm\sqrt{2}$. So, the possible values are $\left(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{2}\right)$.

$f\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right) = 3$ $\quad f\left(-\frac{3}{\sqrt{2}}, \sqrt{2}\right) = -3$

$f\left(\frac{3}{\sqrt{2}}, -\sqrt{2}\right) = -3$ $\quad f\left(-\frac{3}{\sqrt{2}}, -\sqrt{2}\right) = 3$.

So, the max occurs at $\left(\frac{3}{\sqrt{2}}, \sqrt{2}\right)$ and $\left(-\frac{3}{\sqrt{2}}, -\sqrt{2}\right)$, at a value of 3.

1.3 Lagrange Multipliers on Boundaries

This method of Lagrange multiplier can also be used to find maxima and minima on the boundaries of closed and bounded sets. For example, here's how we'd solve the problem we did at the end of section 12.8 using Lagrange multipliers.

Example
Find the global max and min points for the funciton

$$f(x,y) = x^2 - 6x + y^2 - 8y + 7$$
on
$$S = \{(x,y) | x^2 + y^2 \le 1\}$$

using the method of Lagrange multipliers.

$f_x(x,y) = 2x - 6 \quad f_y(x,y) = 2y - 8$

The only stationary point is $(3,4)$, outside of the set S, so the only possible extrema are on the boundary.

$\nabla f(x,y) = \langle 2x-6, 2y-8 \rangle$

$g(x,y) = x^2+y^2-1 \Rightarrow \nabla g(x,y) = \langle 2x, 2y \rangle.$

$\Rightarrow \quad 2x - 6 = \lambda 2x \quad , \quad 2y - 8 = \lambda 2y$

$\Rightarrow \quad x(1-\lambda) = 3 \qquad y(1-\lambda) = 4$

$\Rightarrow \quad 1 - \lambda = \frac{3}{x} \quad$ So, $\; y\left(\frac{3}{x}\right) = 4 \Rightarrow y = \frac{4}{3}x$

$x^2 + \left(\frac{4}{3}x\right)^2 = 1 \Rightarrow \frac{25}{9}x^2 = 1 \Rightarrow x = \pm\frac{3}{5}, \; y = \pm\frac{4}{5}$

$f\left(\frac{3}{5}, \frac{4}{5}\right) = -2 \qquad f\left(-\frac{3}{5}, \frac{4}{5}\right) = \frac{26}{5}$ So, min of -2 at $\left(\frac{3}{5}, \frac{4}{5}\right)$

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = 18 \qquad f\left(\frac{3}{5}, -\frac{4}{5}\right) = \frac{54}{5}$ max of 18 at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

Note: Given $y = \frac{4}{3}x$ only $\left(\frac{3}{5}, \frac{4}{5}\right)$ and $\left(-\frac{3}{5}, -\frac{4}{5}\right)$ were possible.

1.4 Multiple Lagrange Multipliers

Finally, Lagrange multipliers can be used when there are more than two variables, and more than one constraint. For example, when dealing with a function of three variables, $f(x, y, z)$ and two constraints $g(x, y, z) = 0$ and $h(x, y, z) = 0$, the method of Lagrange multipliers states that the extrema will be at the solutions to the equations:

$$\begin{gathered} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) = 0 \\ h(x,y,z) = 0 \end{gathered}$$

Example

Find the minimum distance from the origin to the line of intersection of the two planes:

$$x + y + z = 8 \text{ and } 2x - y + 3z = 28.$$

$d^2(x,y,z) = x^2+y^2+z^2$ $\qquad g(x,y,z) = x+y+z-8$

$\nabla d^2(x,y,z) = \langle 2x, 2y, 2z \rangle$ $\qquad h(x,y,z) = 2x-y+3z-28$

$\nabla g(x,y,z) = \langle 1, 1, 1 \rangle \Rightarrow$ $\qquad 2x = \lambda + 2\mu$

$\nabla h(x,y,z) = \langle 2, -1, 3 \rangle$ $\qquad 2y = \lambda - \mu$

$2z = \lambda + 3\mu$

$x+y+z = 8$ $\quad \cancel{\Rightarrow x + 2x + 3z - 28 + z = 8}$

$2x - y + 3z = 28$ $\quad \cancel{\Rightarrow 3x + 4z = 36}$

$\cancel{2x = \lambda + 2\mu = y + 3z + 28 = \lambda + 2\mu}$ $\quad \cancel{z =}$

$\cancel{2y = \lambda = \mu}$

$\cancel{2z = \lambda + 3\mu}$

$x = 4,\ y = -2,\ z = 6,\ \lambda = 0,\ \mu = 4$

$$\Rightarrow \begin{aligned} 2x \quad\quad -\lambda - 2\mu &= 0 \\ 2y \quad -\lambda + \mu &= 0 \\ 2z - \lambda - 3\mu &= 0 \\ x + y + z &= 8 \\ 2x - y + 3z &= 28 \end{aligned}$$

$\Rightarrow$ So, the distance is:

$\sqrt{4^2 + (-2)^2 + 6^2} = \sqrt{56}$

$= \boxed{2\sqrt{14}}$