Math 2210 - Section 12.8 Notes

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1 Maxima and Minima

1.1 Criteria for Maxima and Minima

If we have a function $f(x, y)$ with a domain S we define:

1. $f(x_0, y_0)$ is a *global maximum value* of f on S if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in S.
2. $f(x_0, y_0)$ is a *global minimum value* of f on S if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in S.
3. $f(x_0, y_0)$ is a *global extreme value* of f on S if it is either a global maximum value or a global minimum value.

Now, a point $(x_0, y_0) \in S$ is a *local* maximum or minimum value if it is a global maximum or minimum on $S \cap N$, where N is any neighborhood of (x_0, y_0) in S. Note for it to be a local maximum or minimum this must be true for *any* neighborhood N.

These are just the definitions you'd expect.

Now, the theory of maxima and minima begins with a theorem that is rather difficult to prove, but intuitively obvious.

Max-Min Existence Theorem - If f is continuous on a closed and bounded set S, then f attains both a (global) maximum value and a (global) minimum value of S.

We wont' be proving this, but it is proven in most texts on advanced calculus. It's a pain to prove.

Now, we want to figure out a criteria for determining which points are local maxima or minima. To aid us in this task, we have another big theorem, this time easier to prove (it's proven in the textbook, but not in these notes) but perhaps less intuitively obvious.

Critical Point Theorem - Let f be defined on a set S containing (x_0, y_0). If $f(x_0, y_0)$ is an extreme value, then (x_0, y_0) must be a critical point; that is, either (x_0, y_0) is:

1. a boundary point of S; or
2. a stationary point of f; or
3. a singular point of f.

Now, recall that a stationary point is a point in S at which the functin f is differentiable and $\nabla f = \mathbf{0}$. A singular point is a point where f is not differentiable.

1.2 Critical Point Test

Now, if we have a critical point (a boundary, stationary, or singular point) then the point may, or may not, be a local maximum or minimum. We'll discuss what to do with boundary points later, and singular points you just have to check individually (we won't be dealing with situations where there are an infinite number of singular points, yuck!), but there's a very useful test for stationary points that is the big brother of the second derivative test from single-variable calculus.

Second Partials Test

Suppose that $f(x, y)$ has continuous second partial derivatives in a neighborhood of (x_0, y_0) and that $\nabla f(x_0, y_0) = \mathbf{0}$. Let

$$D = D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

Then

1. if $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then $f(x_0, y_0)$ is a local maximum value.
2. if $D < 0$ and $f_{xx}(x_0, y_0) > 0$ then $f(x_0, y_0)$ is a local minimum value.
3. if $D < 0$ then $f(x_0, y_0)$ is a saddle point (not an extreme value).
4. if $D = 0$ then the test is inconclusive.

This number D is actually the determinant of a matrix called the "Hessian":

$$H(f) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Example

For $f(x, y) = xy^2 - 6x^2 - 3y^2$, find all critical points. Indicate whether each is a min, max, or saddle point.

$f_x(x,y) = y^2 - 12x \qquad f_x(x,y) = 0 \Rightarrow y^2 = 12x$

$f_y(x,y) = 2xy - 6y \qquad f_y(x,y) = 0 \Rightarrow 2xy = 6y$

If $y \neq 0 \Rightarrow x = 3$ and $y = \pm 6$. So, the critical points are $\{(0,0), (3,6), (3,-6)\}$. $\mathbb{R}^2$ has no boundary and polynomials are everywhere differentiable, so these are the only critical points.

$f_{xx}(x,y) = -12 \Rightarrow D(x,y) = (-12)(2x-6) - (2y)^2 = 72 - 24x - 4y^2$

$f_{yy}(x,y) = 2x - 6 \qquad D(0,0) = 72$

$f_{xy}(x,y) = 2y \qquad D(3,6) = -144$

$D(3,-6) = -144$

So, max of 0 at $(0,0)$. Saddle points at $(3,6)$ and $(3,-6)$.

Example

For $f(x, y) = e^{-(x^2+y^2-4y)}$, find all critical points. Indicate whether each is a min, max, or saddle point.

Again, $f(x,y)$ is differentiable on all $\mathbb{R}^2$ and $\mathbb{R}^2$ has no boundary.

$f_x(x,y) = -2x\, e^{-(x^2+y^2-4y)}$

$f_y(x,y) = (4-2y)\, e^{-(x^2+y^2-4y)}$

$f_x(x,y) = 0 \Rightarrow x = 0$

$f_y(x,y) = 0 \Rightarrow y = 2$

$D = (4x^2-2)((4-2y)^2-2)\, e^{-2(x^2+y^2-4y)} - \left[2x(4-2y)e^{-(x^2+y^2-4y)}\right]^2$

$D(0,2) = 4e^{-8} - 0 = 4/e^8 > 0$

$f_{xx}(0,2) = -2/e^4$

So, $f(0,2) = 1$ is a maximum. In fact, a global maximum.

1.3 Dealing with Boundary Points

Dealing with boundary points can be a major pain, because usually the boundary is a curve, and it's not always easy to parameterize the curve. The basic idea is that to find the maxima and minima on a boundary, you break the boundary down into curves, and then use methods from calculus 1 to find the maxima and minima on those curves. It isn't always obvious, or even remotely obvious, how to do this, and even when it is fairly straightforward, it can still be a major pain. I think the best way to get a feel for how to do this is just to see it in the context of some examples.

Example

Find the global max value and min value for the ~~cone~~ elliptical paraboloid $f(x, y) = x^2 + y^2$ on $S = \{(x, y) | x \in [-1, 3], y \in [-1, 4]\}$ and points that yield these extreme values.

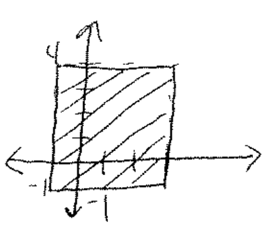
$f_x = 2x$ $\quad f_{xx} = 2$ $\quad f_{xy} = 0$

$f_y = 2y$ $\quad f_{yy} = 2$

In the interior of S there is one stationary point (0, 0), and it is a local minimum.

$D(x,y) = 4$, so $D(0,0) = 4$.

There are 4 boundary segments:



For $y = -1$ we have $f(x) = x^2 + 1$, which on $[-1, 3]$ is maximized for $x = 3$.

$\Rightarrow f(3, -1) = 10$

For $y = 4$ we have $f(x) = x^2 + 16$, which on $[-1, 3]$ is maximized for $x = 3$.

$\Rightarrow f(3, 4) = 25$

For $x = -1$ we have $f(y) = y^2 + 1$, which is maximized for $y = 4$.

$\Rightarrow f(-1, 4) = 17$

For $x = 3$ we have $f(y) = y^2 + 9$, which is maximized for $y = 4$.

$\Rightarrow f(3, 4) = 25$

$\Rightarrow$ Min at (0, 0) ~~At~~ of 0, Max at (3, 4) of 25.

Example
Find the global max and min points for the function

$$f(x, y) = x^2 - 6x + y^2 - 8y + 7$$
on
$$S = \{(x, y) | x^2 + y^2 \leq 1\}.$$

$f_x(x,y) = 2x - 6$ $f_{xx}(x,y) = 2$ $f_{yy}(x,y) = 2$ $f_{xy}(x,y) = 0$
$f_y(x,y) = 2y - 8$

$\nabla f(x,y) = 0$ at $(3,4)$, which is outside of S.

We can parameterize the boundary of S by
$x(t) = \cos(t)$ $y(t) = \sin(t)$ $0 \leq t \leq 2\pi$.

$$f(t) = \cos^2(t) - 6\cos(t) + \sin^2(t) - 8\sin(t) + 7$$
$$= 8 - 6\cos(t) - 8\sin(t)$$
$$f'(t) = 6\sin(t) - 8\cos(t) = 0$$
$$\Rightarrow \tan(t) = 4/3 \quad \Rightarrow t = .927, .927 + \pi$$
$$f''(t) = 6\cos(t) + 8\sin(t)$$
$$f''(.927) = 10, \quad f''(.927+\pi) = -10.$$

So, $t = .927$ is a local min, $t = .927 + \pi$ is a local max

$f(.927) = -2$, $f(.927 + \pi) = 18$.

So, a minimum of -2 at $(\frac{3}{5}, \frac{4}{5})$
A maximum of 18 at $(-\frac{3}{5}, -\frac{4}{5})$.