Math 2210 - Section 12.7 Notes

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Fall 2008

1 Tangent Planes and Approximations

1.1 Implicitly Defined Surfaces and their Tangent Planes

We recall from calculus I, and briefly from our lecture on the chain rule, that we can define curves in $\mathbb{R}^2$ implicitly by an equation of the form $f(x, y) = k$, where k is a constant, or without loss of generality by $f(x, y) = 0$. An example is the unit circle $x^2 + y^2 = 1$, which can also be written as $x^2 + y^2 - 1 = 0$. Now, the same idea applies to surfaces in three dimensions, as we've already seen in our study of quadric surfaces. A surface in $\mathbb{R}^3$ can be defined by an equation of the form $F(x, y, z) = k$, where k is a constant, or again without loss of generality as $F(x, y, z) = 0$. For example the unit sphere is $x^2 + y^2 + z^2 = 1$, which can be rewritten as $x^2 + y^2 + z^2 - 1 = 0$.

Now, if we consider a curve on this surface, $(x(t), y(t), z(t))$, where the component functions all must satisfy the surface equation $F(x, y, z,) = k$ then we get the relation:

$$F(x(t), y(t), z(t)) = k$$

which says that the function F, when viewed as a function of the variable t, is a constant. Well, here we can apply the chain rule to get:

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = \frac{d}{dt}(k) = 0.$$

If we view the curve as a vector-valued function $\mathbf{r}(t) =< x(t), y(t), z(t) >$ then we see that this is equivalent to:

$$\nabla F \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Now, $\frac{d\mathbf{r}}{dt}$ is tangent to the curve $\mathbf{r}(t)$, and as the above equation is valid for any curve on the surface, we see that the gradient vector of the function F at a point on the surface is perpendicular to all tangent lines to the surface at that point. This provides the following definition:

Definition - If a surface is defined implicitly as $F(x, y, z) = k$ then if F is differentiable at a point $P(x_0, y_0, z_0)$ on the surface, with $\nabla F(x_0, y_0, z_0) \neq 0$ then the plane through P perpendicular to $\nabla F(x_0, y_0, z_0)$ is called the *tangent plane* to the surface at the point P.

Now, the equation for this tangent plane is:

1. For a surface defined as $F(x, y, z) = k$:

   $$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

   which just comes from the above discussion and our earlier equations about tangent planes.

2. For a surface defined as $z = f(x, y)$:

   $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

   which is the equation we derived earlier in this chapter, but it can also be derived from the previous equation with $F(x, y, z) = f(x, y) - z = 0$.

Example
Find the equation of the tangent plane to $8x^2+y^2+8z^2 = 16$ at $(1, 2, \sqrt{2}/2)$.

$F(x, y, z) = 8x^2 + y^2 + 8z^2$

$\nabla F(x, y, z) = \langle 16x, 2y, 16z \rangle$

$\nabla F(1, 2, \sqrt{2}/2) = \langle 16, 4, 8\sqrt{2} \rangle$

Tangent plane:
$16(x-1) + 4(y-2) + 8\sqrt{2}(z - \sqrt{2}/2) = 0$
$\Rightarrow \boxed{16x + 4y + 8\sqrt{2}\,z = 32}$ or $4x + y + 2\sqrt{2}\,z = 8$

1.2 Differentials and Approximations

If we have a surface defined as $z = f(x, y)$ and if (x_0, y_0, z_0) is a fixed point on the surface, we can introduce a new coordinate system (dx, dy, dz) parallel to the old coordinate system, but with (x_0, y_0, z_0) as the origin. (In other words, we just change (x, y, z) to (dx, dy, dz) and change (x_0, y_0, z_0) to $(0, 0, 0)$). If we make this change then our equation for the tangent plane to the surface at the point (x_0, y_0, z_0) changes from:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

to:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

which is much simpler. We can actually use this to define a quantity dz, which we call the *differential* of the function $f(x, y)$ at the point (x_0, y_0).

Definition - Let $z = f(x, y)$, where f is a differentiable function, and let dx and dy (called the differentials of x and y) be variables. The *differential* of the dependent variable, dz, also called the *total differential* of f and written $df(x, y)$, is defined as:

$$dz = df(x, y) = f_x(x, y)dx + f_y(x, y)dy = \nabla f \cdot < dx, dy >.$$

This differential derives from the concept of a tangent plane, and how for a function that is differentiable at a point the tangent plane is a good approximation of the function around that point. Consequently, we can use the differential dz to obtain an estimate for the change in the function Δz when we change x and y by a small amount.

Example - Use dz to approximate the change in z as (x, y) moves from $(2, 3)$ to $(2.03, 2.98)$ for $z = x^2 - 5xy + y$. Compare this estimate to the actual change in value.

$f_x(x, y) = 2x - 5y \qquad f_x(2,3) = -11$

$f_y(x, y) = -5x + 1 \qquad f_y(2,3) = -9$

$dz = -11(.03) - 9(-.02) = -.33 + .18 = \boxed{-.15}$

Actual change:

$(-23.1461) - (-23) = \boxed{-.1461}$

$\left| \frac{-.1461 - -.15}{-.1461} \right| \times 100\% = \boxed{2.67\% \text{ off}}$

1.3 Taylor Polynomials for Multivariable Functions

We won't go too deep into Taylor polynomials for multivariable functions because, to be honest, they just get too big and too nasty too fast, and are best left to computers. However, the idea is basically the same as the idea for single variable functions.

For a single variable function we can get an approximation for the function by using its first derivative and finding a tangent line:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

but we can get a better approximation using an approximating parabola, whose equation we get from using the second derivative:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2.$$

Well, the same idea applies to surfaces. We can get an approximation with a tangent plane, but we can get a better approximation with a "tangent" elliptic paraboloid. The equation is:

$$f(x, y) \approx f(x_0, y_0) + [f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)] +$$
$$\frac{1}{2}[f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2]$$

Now, this can extend to third or higher-order approximations, and we can further extend this to functions of three or more variables, but as I said, it gets messy, and is best left to a computer.

Example - For the function $f(x,y) = \tan\left(\frac{x^2+y^2}{64}\right)$ find the second order Taylor polynomial based at $(0,0)$.

$$f_x = \frac{x}{32}\sec^2\left(\frac{x^2+y^2}{64}\right)$$

$$f_y = \frac{y}{32}\sec^2\left(\frac{x^2+y^2}{64}\right)$$

$$f_{xx} = \frac{x}{32}\left(2\sec^2\left(\frac{x^2+y^2}{64}\right)\tan\left(\frac{x^2+y^2}{64}\right)\right)\left(\frac{x}{32}\right)$$

$$= \frac{x^2}{512}\sec^2\left(\frac{x^2+y^2}{64}\right)\tan\left(\frac{x^2+y^2}{64}\right) + \frac{1}{32}\sec^2\left(\frac{x^2+y^2}{64}\right)$$

$$f_{yy} = \frac{y^2}{512}\sec^2\left(\frac{x^2+y^2}{64}\right)\tan\left(\frac{x^2+y^2}{64}\right) + \frac{1}{32}\sec^2\left(\frac{x^2+y^2}{64}\right)$$

$$f_{xy} = \frac{xy}{512}\sec^2\left(\frac{x^2+y^2}{64}\right)\tan\left(\frac{x^2+y^2}{64}\right)$$

So, the second order Taylor polynomial is:

$$f(x,y) \approx 0 + 0 + \frac{1}{2}\left[\frac{1}{32}x^2 + \frac{1}{32}y^2\right]$$

$$= \boxed{\frac{x^2+y^2}{64}}$$