Math 2210 - Section 12.6 Notes

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1 The Chain Rule

1.1 The Calculus I Chain Rule

In calculus I we learned that if we have a composite of two functions, $y(x) = f(g(x))$ then the derivative of the composite was the derivative of the outside function, multiplied by the derivative of the inside function:

$$y'(x) = f'(g(x))g'(x).$$

Example
What is the derivative of $\ln(\sin(x^2 + e^x))$?

$$\frac{d}{dx}(\ln(\sin(x^2+e^x))) = \frac{1}{\sin(x^2+e^x)} \cdot \cos(x^2+e^x) \cdot (2x+e^x)$$

$$= \boxed{(2x+e^x)\cot(x^2+e^x)}$$

1.1.1 The First Version of the Multivariable Chain Rule

If $z = f(x, y)$ is a function of two variables, and both of those variables are in turn functions of a single parameter t, then we can view the function z as a function of the single parameter t.

The idea behind this sentence is much easier to understand than it appears. For example, suppose we have the function $z = \sin(x + y)$, with $x = t^2$ and $y = t^3$, then we could write z as a function of just t, namely $z = \sin(t^2 + t^3)$.

Well, z when expressed like this is just a single variable function, and so if the functions f, x, and y are differentiable, then it makes sense to talk about the derivative of z with respect to t. The relationship between the derivative of z with respect to t, and the other derivatives of f, x, and y are:

Theorem

Let $x = x(t)$ and $y = y(t)$ be differentiable at t, and let $z = f(x, y)$ be differentiable at $(x(t), y(t))$. Then $z = f(x(t), y(t))$ is differentiable at t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

This is the first version of the chain rule for multivariable functions. Basically, it's just saying that the amount that z changes when we change t is how much z changes when we change x, multiplied by how much x changes when we change t added to how much z changes when we change y, multiplied by how much y changes when we change t. Again, that's a long sentence, but walk through it and you'll see it's really just logic. The proof is pretty straightforward.

Proof

If we simplify notation and let $\mathbf{p} = (\Delta x, \Delta y)$, and $\Delta z = f(\mathbf{p}+\Delta\mathbf{p}) - f(\mathbf{p})$ then since f is differentiable we have:

$$\begin{aligned}\Delta z = f(\mathbf{p} + \Delta\mathbf{p}) - f(\mathbf{p}) &= \nabla f(\mathbf{p}) \cdot \Delta\mathbf{p} + \epsilon(\mathbf{p}) \cdot \Delta\mathbf{p} \\ &= f_x(\mathbf{p})\Delta x + f_y(\mathbf{p})\Delta y + \epsilon(\Delta\mathbf{p}) \cdot \Delta\mathbf{p}\end{aligned}$$

where $\epsilon(\mathbf{p}) \to \mathbf{0}$ as $\Delta\mathbf{p} \to \mathbf{0}$.

Now, if we divide both sides by Δt and take the limit as $\Delta t \to 0$ we get:

$$\frac{dz}{dt} = f_x(\mathbf{p})\frac{dx}{dt} + f_y(\mathbf{p})\frac{dy}{dt}.$$

which is what we want to prove.

Example

Find $\frac{dw}{dt}$ given $w = x^2y - y^2x$, $x = \cos t$, $y = \sin t$.

$$\frac{\partial w}{\partial x} = 2xy - y^2 \qquad \frac{dx}{dt} = -\sin(t)$$

$$\frac{\partial w}{\partial y} = x^2 - 2xy \qquad \frac{dy}{dt} = \cos(t)$$

$$\frac{\partial w}{\partial x} = 2(\cos t)(\sin t) - \sin^2 t \Rightarrow \frac{dw}{dt} = [2\cos t\sin t - \sin^2 t](-\sin t)$$

$$+ [\cos^2 t - 2\cos t\sin t]\cos t$$

$$\frac{\partial w}{\partial y} = \cos^2 t - 2\cos t\sin t \qquad = \boxed{\sin^3 t + \cos^3 t - 2\sin^2 t\cos t - 2\sin t\cos^2 t}$$

1.2 The Second Version of the Multivariable Chain Rule

This is a natural extension of the concepts we just discussed. Suppose that we have a function $z = f(x, y)$ and x and y are themselves functions of two other parameters s and t, say $x = x(s, t)$ and $y = y(s, t)$. Then z itself can be viewed as a function of s and t, and if everything is differentiable we can takes its partial derivative with respect to s or t. The corresponding relations are:

Theorem - Let $x = x(s, t)$ and $y = y(s, t)$ have first partial derivaties at (s, t) and let $z = f(x, y)$ be differentaible at $(x(s, t), y(s, t))$. Then $z = f(x(s, t), y(s, t))$ has first partial derivatives given by:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Example
Find $\frac{\partial w}{\partial t}$ given $w = \ln(x+y) - \ln(x-y)$ with $x = te^s$ and $y = e^{st}$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \left[\left(\frac{1}{x+y}\right) - \left(\frac{1}{x-y}\right)\right] e^s$$
$$+ \left[\left(\frac{1}{x+y}\right) + \left(\frac{1}{x-y}\right)\right] s e^{st}$$
$$= \frac{e^s}{te^s + e^{st}} - \frac{e^s}{te^s - e^{st}} + \frac{se^{st}}{te^s + e^{st}} + \frac{se^{st}}{te^s - e^{st}}$$
$$= \boxed{\frac{e^s + se^{st}}{te^s + e^{st}} + \frac{se^{st} - e^s}{te^s - e^{st}}}$$

We note that we can naturally extend these ideas to functions of three or more dimensions.

Example
If $w = x^2 + y^2 + z^2 + xy$, where $x = st$, $y = s - t$ and $z = s + 2t$ calculate $\frac{\partial w}{\partial t}$.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial w}{\partial x} = 2x + y = 2st + s - t \qquad \frac{\partial x}{\partial t} = s$$

$$\frac{\partial w}{\partial y} = 2y + x = 2s - 2t + st \qquad \frac{\partial y}{\partial t} = -1$$

$$\frac{\partial w}{\partial z} = 2z = 2(s+2t) = 2s + 4t \qquad \frac{\partial z}{\partial t} = 2$$

$$\Rightarrow \frac{\partial w}{\partial t} = (2st + s - t)s + (2s - 2t + st)(-1) + (2s + 4t)2$$
$$= 2s^2t + s^2 - st - 2s + 2t - st + 4s + 8t$$
$$= \boxed{2s^2t + s^2 - 2st + 2s + 10t}$$

1.3 The Implicit Function Theorem

We may remember from calculus I that it is possible to define a curve implicitly as all points x and y that satisfy a given relation $F(x,y) = 0$. The unit circle, for example, would be a curve of this form: $x^2 + y^2 - 1 = 0$. This is a more general concept that a function $y = f(x)$, in that neither of the variables must be a function of the other one.

It is possible to talk about the slope of a curve defined in this way around a point on the curve. We learned in calculus I a rather long and laborious way of solving this type of problem. Here we'll learn a short cut.

If we have a relation $F(x,y) = 0$ then if we differentiate both sides with respect to x we get:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Now, if we note that $\frac{dx}{dx} = 1$ then after some algebra we get:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

which makes implicit differentiation much easier.

Example

For the curve defined by $F(x,y) = x^3 + x^2y - 10y^4$ $= 0$ calculate $\frac{dy}{dx}$ as a function of x and y.

Calculus I way:

$$3x^2(dx) + 2xy\,dx + x^2dy - 40y^3dy = 0$$

$$\Rightarrow \boxed{\frac{3x^2+2xy}{40y^3-x^2} = \frac{dy}{dx}}$$

Implicit Function Theorem:

$$\frac{\partial F}{\partial x} = 3x^2+2xy$$

$$\frac{\partial F}{\partial y} = -40y^3+x^2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} = \boxed{\frac{3x^2+2xy}{40y^3-x^2}}$$

We also get similar relations for surfaces defined by $F(x, y, z) = 0$.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \text{ and } \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

Example

If $F(x, y, z) = x^3 e^{y+z} - y \sin(x - z) = 0$ defines z implicitly as a function of x and y, find $\dfrac{\partial z}{\partial x}$.

$$\frac{\partial F}{\partial x} = 3x^2 e^{y+z} - y\cos(x-z)$$

$$\frac{\partial F}{\partial z} = x^3 e^{y+z} + y\cos(x-z)$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{y\cos(x-z) - 3x^2 e^{y+z}}{y\cos(x-z) + x^3 e^{y+z}}}$$