

Math 2210 - Section 12.5 Notes

Dylan Zwick

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1 Directional Derivatives and Gradients

So far we've learned about partial derivatives with respect to x and y , and also the general notion of differentiability. We now want to extend these notions so that we can deal with partial derivatives in any direction.

If we have a function $f(x, y)$ this usually describes a surface $z = f(x, y)$. We can imagine this surface as a mountain. If you're standing on a mountain and I ask you what the slope is at that point, you'd have to tell me that you can't answer the question. The reason for this is that the slope is different depending on which direction you walk. If you walk straight up the mountain the slope is probably quite large, while if you walk along a trail the slope is usually much less. So, we can't really talk about *the* slope at a point, only the slope in a given direction at that point. This is the idea behind the directional derivative, and it's this concept that we will formalize in this section.

1.1 The Directional Derivative

1.1.1 Definition

If we have a function $f(x, y)$ we can view the input (x, y) as coming from a position vector \mathbf{p} that points from the origin to the point (x, y) . So, we can view the function $f(x, y)$ as a function of the vector \mathbf{p} . Using this idea we can represent the partial derivative with respect to x as:

$$f_x(\mathbf{p}) = \lim_{h \rightarrow 0} \frac{f(\mathbf{p} + h\mathbf{i}) - f(\mathbf{p})}{h}$$

Along these lines we can define the directional derivative in any direction to be:

Definition - For a unit vector \mathbf{u} the *directional derivative* of the function f at \mathbf{p} in the direction of \mathbf{u} is defined as:

$$D_{\mathbf{u}}(f(\mathbf{p})) = \lim_{h \rightarrow 0} \frac{f(\mathbf{p} + h\mathbf{u}) - f(\mathbf{p})}{h}$$

We note something very important in this definition is that the vector \mathbf{u} must be a *unit* vector. This is often something that trips people up who see this for the first time.

1.1.2 The Directional Derivative and the Gradient

We recall that for a function $f(x, y)$, a.k.a. $f(\mathbf{p})$, that the gradient of the function was defined as the vector:

$$\nabla f(\mathbf{p}) = f_x(\mathbf{p})\mathbf{i} + f_y(\mathbf{p})\mathbf{j}$$

Well, the nice thing about taking directional derivatives is that, as you might expect, we don't need to always refer back to the limit definition, and in fact once we've calculated the gradient at a given point \mathbf{p} calculating the directional derivative in any direction \mathbf{u} is easy.

Theorem - Let f be differentiable at \mathbf{p} . Then f has a directional derivative at \mathbf{p} in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p})$$

which can be written more explicitly as:

$$D_{\mathbf{u}}f(x, y) = u_1f_x(x, y) + u_2f_y(x, y)$$

We will not have time to prove this in class, but the proof is straightforward and is in the textbook.

Example - Find the directional derivative of the function $f(x, y) = y^2 \ln x$ at the point $(1, 4)$ in the direction of $\mathbf{a} = \mathbf{i} - \mathbf{j}$.

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j}$$

$$\nabla f(x, y) = \frac{y^2}{x} \hat{i} + 2y \ln x \hat{j}$$

$$\nabla f(1, 4) = 16 \hat{i} + 0 \hat{j}$$

$$\nabla f(1, 4) \cdot \vec{u} = \boxed{\frac{16}{\sqrt{2}}}$$

Example - Find the directional derivative of the function $f(x, y) = 2x^2 \sin y + yx$ at the point $(1, \pi/2)$ in the direction of $2\mathbf{i} + \mathbf{j}$.

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{2}{\sqrt{5}} \hat{i} + \frac{1}{\sqrt{5}} \hat{j}$$

$$f_x(1, \frac{\pi}{2}) = 4x \sin y + y \Big|_{(1, \pi/2)} = 4 + \frac{\pi}{2}$$

$$f_y(1, \frac{\pi}{2}) = 2x^2 \cos y + x \Big|_{(1, \pi/2)} = 1$$

$$\nabla f(1, \frac{\pi}{2}) \cdot \vec{u} = \frac{8 + \pi}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \boxed{\frac{9 + \pi}{\sqrt{5}}}$$

1.1.3 Direction of Maximum Rate of Change

For a given function f at a point \mathbf{p} a natural question we may wish to ask is in what direction \mathbf{u} will the directional derivative be maximized. In other words, if you're standing on a mountain, in what direction would you have the steepest ascent. The answer to this can be derived pretty easily from the earlier theorem, and it's so straightforward that we'll go through it's proof:

$$D_{\mathbf{u}}f(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p}) = \|\mathbf{u}\| \|\nabla f(\mathbf{p})\| \cos \theta = \|\nabla f(\mathbf{p})\| \cos \theta$$

Now, $\cos \theta$ is maximized (and equal to 1) when $\theta = 0$, and minimized (equal to -1) when $\theta = \pi$. What this means is that the gradient vector points in the direction of maximum increase, and points away from the direction of maximum decrease. The magnitude of this maximum increase is $\|\nabla f(\mathbf{p})\|$, while the magnitude of the maximum decrease is $-\|\nabla f(\mathbf{p})\|$.

1.2 Level Curves and Gradients

We recall that for a function $z = f(x, y)$ the level curves for a given constant k are all the input values that have an output value k . In other words, all points (x, y) such that $f(x, y) = k$. These sets of points usually form curves. Now, we note that if we move along one of these level curves then, by definition, $f(x, y)$ is not changing, and so it must be that along these level curves the directional derivative is 0. This means that the angle between the direction of the level curve at a point and the direction of the gradient vector at that point must be $\pi/2$. In still other words the level curves are perpendicular to the gradient vector field.

Theorem - The gradient of f at a point P is perpendicular to the level curve of f that goes through P .

Example - Find the vector in the direction of most rapid increase for the function $f(x, y) = e^y \sin x$ at the point $(5\pi/6, 0)$. Then find the rate of change in that direction.

$$f_x(x, y) = e^y \cos x \quad f_x\left(\frac{5\pi}{6}, 0\right) = -\frac{\sqrt{3}}{2}$$

$$f_y(x, y) = e^y \sin x \quad f_y\left(\frac{5\pi}{6}, 0\right) = \frac{1}{2}$$

$$\nabla f\left(\frac{5\pi}{6}, 0\right) = -\frac{\sqrt{3}}{2} \hat{i} + \frac{1}{2} \hat{j} \quad \|\nabla f\left(\frac{5\pi}{6}, 0\right)\| = 1$$

\Rightarrow Direction is $\left\langle -\frac{\sqrt{3}}{2}, 1 \right\rangle$ rate is $\boxed{1}$

1.2.1 Higher Dimensions

We note here finally that just as functions of two variables $f(x, y)$ give us level curves, functions of three variables $f(x, y, z)$ give us level surfaces, although these can be harder to visualize for obvious reasons. As an example, the surface defined by:

$$w = f(x, y, z) = x^2 + y^2 - z^2$$

would have a level surface for $w = 1$ that would be a hyperboloid of one sheet.

Example - Graph the level surface just described.

